

SOLUTIONS

Midterm 2

Math 1181H - Honors Calculus (Section 110)

Prof. Cueto

Friday Oct. 20th 2017

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
 - a) get the **correct answer**, and
 - b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- You will find a sheet of Scratch paper at the end. You can take it out if you wish.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five (55) minutes** to complete the exam.
- Do not forget to write your full name (in PRINT) in the space provided below and on the bottom of the last page.

Full Name (Print): _____

Good luck!

Exercise 1. [9 points]

a) Compute the following:

$$(1) \lim_{x \rightarrow 0} \frac{\tan^2(4x)}{x^2}$$

$$\begin{aligned} &= \left(\lim_{x \rightarrow 0} \frac{\tan(4x)}{x} \right)^2 \\ &= \left(\lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{4x}}{\frac{4}{\cos 4x}} \right)^2 \\ &\quad \downarrow x \rightarrow 0 \quad \downarrow \cos(0)=1 \\ &= 4^2 = \boxed{16} \end{aligned}$$

$$(2) \int (2x+5)^{299} dx.$$

Substitution

$$u = 2x+5$$

$$du = 2dx$$

$$\int (2x+5)^{299} dx$$

$$= \int u^{299} \frac{du}{2}$$

$$= \frac{1}{2} \frac{u^{300}}{300} + C$$

$$= \boxed{\frac{(2x+5)^{300}}{600} + C}$$

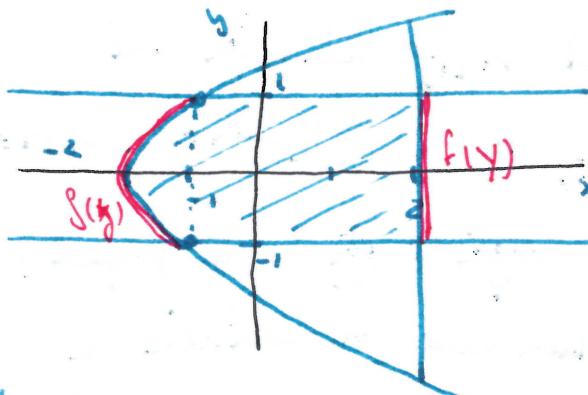
$$b) \text{ Compute } \frac{d}{dx} \int_0^{x^6} \frac{\cos(t) dt}{\sqrt{1 + \sin^2(t)}}. \quad f'(x)$$

$$\text{Use the FTC. } g(u) = \int_0^u \frac{\cos(t) dt}{\sqrt{1 + \sin^2(t)}}$$

$$\text{so we get. } g'(u) = \frac{\cos(u)}{\sqrt{1 + \sin^2(u)}} \Rightarrow f(x) = g(x^6) \quad \text{we get } f' \text{ by}$$

$$\text{the Chain Rule. } f' = g'(x^6) \cdot (x^6)' = \boxed{\frac{\cos(x^6)}{\sqrt{1 + \sin^2(x^6)}} \cdot 6x^5}$$

Exercise 2. [10 points] Sketch the region bounded by the curves $x = y^2 - 2$, $x = 2$, $y = 1$, and $y = -1$ and compute its area.



Need to find where $x = y^2 - 2$ meets $y = 1$ & $y = -1$.

$$x = y^2 - 2 \text{ & } y = 1 \text{ gives } x = 1^2 - 2 = -1$$

$$x = y^2 - 2 \text{ & } y = -1 \text{ gives } x = -1$$

It's best to integrate along the y-variable.

$$A = \int_{-1}^1 (f(y) - g(y)) dy = 2 \int_0^1 f(y) - g(y) dy.$$

By symmetry

$$f(y) = 2, \quad g(y) = y^2 - 2$$

$$A = 2 \int_0^1 (2 - (y^2 - 2)) dy = 2 \int_0^1 (4 - y^2) dy = 2 \left(4y - \frac{y^3}{3} \right) \Big|_0^1$$

$$= 2 \left(4 - \frac{1}{3} \right) = \frac{8 - \frac{2}{3}}{3} = \frac{24 - 2}{3} = \boxed{\frac{22}{3}}$$

Exercise 3. [9 points] Find the amplitude and frequency of the simple harmonic motion of a particle with trajectory $x(t) = 4 \sin(4t) + 3 \cos(4t)$. Find its maximal velocity.

We know by instruction that $x(t)$ satisfies the diff'! eqn: $x''(t) + 4^2 x(t) = 0$ ($\sin(4t) \& \cos(4t)$ do)

so $a=4$ and Period $T = \frac{2\pi}{4} = \frac{\pi}{2}$ & frequency $= \frac{1}{T} = \frac{2}{\pi}$

The trajectory of the (SHM) can also be written as

$$x(t) = A \sin(4t + b).$$

We evaluate at convenient points to find the amplitude $|A|$.

$$t=0: x(0) = A \sin(b) = 4 \cdot \sin(0) + 3 \cos(0) = 3$$

$$t=\frac{\pi}{8} \quad A \left(\frac{\pi}{8}\right) = A \sin\left(\frac{\pi}{2} + b\right) = \underbrace{4 \sin\left(\frac{\pi}{2}\right)}_{=1} + \underbrace{3 \cos\left(\frac{\pi}{2}\right)}_{=0} = 4$$

$$\text{Byt } \sin\left(\frac{\pi}{2} + b\right) = \underbrace{\sin \frac{\pi}{2} \cos b}_{=1} + \sin b \underbrace{\cos \frac{\pi}{2}}_{=0} = \cos b.$$

We get $\begin{cases} A \sin b = 3 \\ A \cos b = 4 \end{cases}$ \rightsquigarrow  so $A^2 = 3^2 + 4^2 = 25$
so $|A| = 5$.

$$\text{Alternative: } \begin{aligned} A^2 \sin^2 b &= 3^2 \\ + A^2 \cos^2 b &= 4^2 \end{aligned}$$

$$A^2 = A^2 (\underbrace{\sin^2 b + \cos^2 b}_{=1}) = 25 \quad \text{so } |A| = 5$$

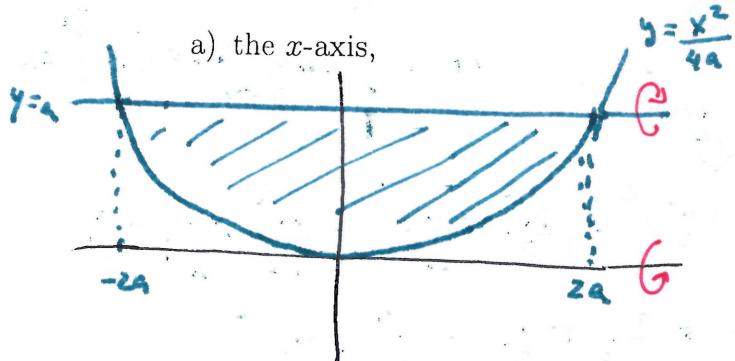
$$\text{Maximal velocity? } x'(t) = 4A \cos(4t + b)$$

$$\text{If } A \geq 5 \quad \cos(4t+b) \text{ has max value } = 1 \quad \text{and } x'(t) \text{ has max}$$

$$\text{If } A \leq -5 \quad \text{min value } = -1 \quad \text{and value } 4 \cdot 5 = 20$$

$$\boxed{\text{Max velocity} = 20}$$

Exercise 4. [12 points] Fix $a > 0$. Consider the area bounded by the curves $x^2 = 4ay$, $y = a$ and $x = 0$. Draw a sketch of this region and find the volumes generated by revolving the area about:



a), the x -axis,

b) the line $y = a$.

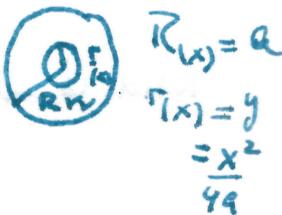
We find the intersection points.

$$x^2 = 4ay \Rightarrow 4a^2 = 4a^2 \Rightarrow (2a)^2$$

$$\text{so } \begin{cases} x = \pm 2a \\ y = a \end{cases} \text{ are the 2 intercepts.}$$

(a) ~~Washer Method~~ \rightsquigarrow WASHER METHOD symmetry \times -Cross Sections

$$\begin{aligned} \text{So Vol} &= \int_{-2a}^{2a} \pi R_{(x)}^2 - \pi r_{(x)}^2 dx = 2 \int_0^{2a} \pi (R_{(x)}^2 - r_{(x)}^2) dx \\ &= 2\pi \int_0^{2a} a^2 - \frac{x^4}{16a^2} dx = 2\pi \left(a^2 x - \frac{x^5}{80a^2} \right) \Big|_0^{2a} \\ &= 2\pi \left(a^2 (2a) - \frac{(2a)^5}{80a^2} \right) = 2\pi a^3 \left(2 - \frac{2^5}{80} \right) = 2\pi a^3 \left(2 - \frac{2}{5} \right) = \boxed{\frac{16\pi a^3}{5}} \end{aligned}$$



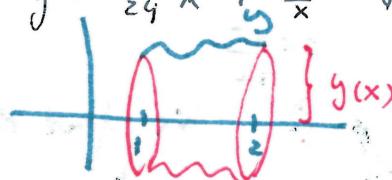
$$\begin{aligned} R_{(x)} &= a \\ r_{(x)} &= y \\ &= \frac{x^2}{4a} \end{aligned}$$

(b) ~~Disk Method~~ \rightsquigarrow DISK METHOD! Radius = $r_{(x)} = a - y = a - \frac{x^2}{4a}$

$$\begin{aligned} \text{Vol} &= 2 \int_0^{2a} \pi r_{(x)}^2 dx = 2\pi \int_0^{2a} \left(a - \frac{x^2}{4a} \right)^2 dx = 2\pi \int_0^{2a} \left(a^2 + \frac{x^4}{16a^2} - \frac{x^2}{2} \right) dx \\ &= 2\pi \left(a^2 x + \frac{x^5}{80a^2} - \frac{x^3}{6} \right) \Big|_0^{2a} = 2\pi \left(a^2 (2a) + \frac{(2a)^5}{80a^2} - \frac{(2a)^3}{6} \right) \\ &= 2\pi a^3 \left(2 + \frac{2}{5} - \frac{8}{3} \right) = 2\pi a^3 \left(\frac{30 + 6 - 80}{15} \right) = 2\pi a^3 \frac{16}{15} \\ &= \boxed{\frac{32\pi a^3}{15}} \end{aligned}$$

Exercise 5. [10 points] Find the surface area of the surface of revolution generated by the curve $y = \frac{1}{24}x^3 + \frac{2}{x}$ with $1 \leq x \leq 2$.

Note: $y > 0$ in the range so



$$A = \int_1^2 2\pi y_{(x)} \sqrt{1+(y'(x))^2} dx$$

Need to compute $y' = \frac{3x^2}{24} - \frac{2}{x^2} \Rightarrow (y')^2 = \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2 = \frac{x^4}{64} + \frac{4}{x^4} - \frac{4}{8}$

$$\text{So } 1+(y')^2 = 1 + \frac{x^4}{64} + \frac{4}{x^4} - \frac{4}{8} = \frac{x^4}{64} + \frac{4}{x^4} + \frac{4}{8} = \left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2$$

Need to compute $\sqrt{1+(y')^2} = \sqrt{\frac{x^2}{8} + \frac{2}{x^2}} = \frac{x^2}{8} + \frac{2}{x^2}$.

$$\begin{aligned} \text{So } A &= \int_1^2 2\pi \cdot \left(\frac{1}{24}x^3 + \frac{2}{x}\right) \left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = \int_1^2 2\pi \left(\frac{x^5}{24 \cdot 8} + \frac{2}{24}x + \frac{2x}{8} + \frac{4}{x^3}\right) dx \\ &= 2\pi \left[\frac{x^6}{24 \cdot 8} + x \left(\frac{1}{12} + \frac{1}{4}\right) + \frac{4}{x^2} \right]_1^2 = 2\pi \left(\frac{x^6}{64 \cdot 8} + \frac{x^2}{6} - \frac{4}{3x^2} \right) \Big|_1^2 \\ &= 2\pi \left(\frac{2^6}{64 \cdot 8} + \frac{2^2}{6} - \frac{4}{2^2} - \left(\frac{1}{64 \cdot 8} + \frac{1}{6} - 2 \right) \right) \\ &= 2\pi \left(\frac{2^6}{64 \cdot 8} + \frac{4}{6} - \frac{1}{2} - \frac{1}{64 \cdot 8} + \frac{1}{6} + 2 \right) = 2\pi \left(\frac{3}{9.2} + 2 + \frac{1}{2} \right) \\ &= 2\pi \left(\frac{7+256}{64} \right) = \pi \left(\frac{7+256}{64} \right) = \boxed{\frac{263}{64}\pi} \end{aligned}$$

For Grader's use only:

1		2	3	4		5	TOTAL
a	b			a	b		