SOLUTIONS

Midterm 3

Math 1181H - Honors Calculus (Section 110)

Prof. Cueto

Tuesday Nov. 21st 2017

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- You will find a sheet of Scratch paper at the end. You can take it out if you wish.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five (55) minutes** to complete the exam.
- Do not forget to write your full name (in PRINT) in the space provided below.

Full Name (Print): _____

Good luck and GO BUCKS!

Exercise 1. **[10 points]** Compute the following limits:

a) $\lim_{x\to\pi} \dfrac{\sin^2(x)}{1+\cos(3\,x)}$ Note the limit is of the form $\frac{0}{0}$, so we use L'Hosp.

$$= \lim_{x\to\pi} \frac{+2\sin x\cos x}{-3\sin(3x)} = -\frac{2}{3}\lim_{x\to\pi}\frac{\sin x}{\sin 3x}\cos x \to -1$$

$\sim 0/0$

$$= \frac{2}{3}\lim_{x\to\pi}\frac{\sin x}{\sin 3x} \underset{\text{L'Hosp. again}}{=} \frac{2}{3}\lim_{x\to\pi}\frac{\cos x}{3\cos 3x} = \frac{2}{9}\frac{\cos\pi}{\cos 3\pi}$$

$0/0$

$$= \frac{2}{9}\frac{-1}{-1} = \boxed{\frac{2}{9}}$$

b) $\lim_{x\to 1^+} (x^2-1)^{x-1}$ This limit is of the form 0^0, so indeterminate

Note $(x^2-1) = \underset{\downarrow\, 0^+}{(x-1)}\,\underset{\downarrow\, 2}{(x+1)}$ so $\lim_{x\to 1^+}(x^2-1)^{x-1} = \lim_{x\to 1^+}(x-1)^{x-1}\cdot\lim_{x\to 1^+}(x+1)^{x-1}$

$$= \lim_{x\to 1^+}(x-1)^{x-1}\cdot(2^0) = \lim_{x\to 1^+}(x-1)^{x-1} \underset{u=x-1\to 0^+}{=} \lim_{u\to 0^+} u^u.$$

Use L'Hospital since it's an indet. 0^0. First, apply ln.

$$\ln\left(\lim_{u\to 0^+} u^u\right) \underset{\ln \text{ cont}}{=} \lim_{u\to 0^+}\ln(u^u) = \lim_{u\to 0^+} u\ln u = \lim_{u\to 0^+}\frac{\ln u}{1/u}$$

$\frac{-\infty}{+\infty}$

$$\underset{\text{L'Hosp.}}{=} \lim_{u\to 0^+}\frac{1/u}{\frac{-1}{u^2}} = \lim_{u\to 0^+}\frac{-u^2}{u} = \lim_{u\to 0^+} -u = 0.$$

So $\lim_{u\to 0^+} u^u = e^0$ & so $\lim_{x\to 1^+}(x^2-1)^{x-1} = 1\cdot 1 = \boxed{1}$.

Exercise 2. **[10 points]** Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Be sure to state which test you are using and verify the conditions.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+3}$

Start w/ abs. convergent. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+3}$

Compare with $\frac{1}{\sqrt{n}}$. $\displaystyle\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n}{n+3} = 1 \neq 0$.

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a p-series with $p=\frac{1}{2}$, it diverges

So does $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+3}$ $\rightsquigarrow$ <u>NOT</u> abs-convergent.

<u>Check</u> convergence with Alternating Series Test $\displaystyle\sum_{n=1} a_n = \frac{\sqrt{n}}{n+3} = f(n)$

for $f(x) = \dfrac{\sqrt{x}}{x+3}$. $f \geq 0$ for $x \geq 1$.

$f \xrightarrow[n\to\infty]{} 0$ (by comparison with $\frac{1}{\sqrt{n}}$).

$$f' = \frac{\frac{1}{2\sqrt{x}}(x+3) - \sqrt{x}}{(x+3)^2} = \frac{x+3-2x}{\sqrt{x}\,(x+3)^2} = \frac{3-x}{\sqrt{x}\,(x+3)^2} < 0 \text{ for } x \geq 3$$

So a_n is decreasing for $n \geq 3$

By AST, the series is <u>convergent</u>

b) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\sin^2(n)}{n^{3/2}}$ almost a. Check abs. conv. $\displaystyle\sum_{n=3}^{\infty}\frac{\operatorname{sen}^2(n)}{n^{3/2}} \leq \sum_{n=3}^{\infty}\frac{1}{n^{3/2}}$.

This is a p-series with $p = \frac{3}{2} > 1$, so it converges! (*)

Since <u>abs. convergent</u> implies convergent, we are done.

(*) $n^{3/2} \geq n^{3/2}$ for $n \geq 3$

$\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^{3/2}} \leq \sum_{n=3}^{\infty}\frac{1}{n^{3/2}}$ $\rightsquigarrow$ p-series.

[<u>Alternatively</u>: $\displaystyle\sqrt[n]{\frac{1}{n^{3/2}}} = \frac{1}{n^{1/2}} \xrightarrow[n\to\infty]{} 0 < 1$ so by the Root Test, we get $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^{3/2}}$ converges.] By comparison, so does $\displaystyle\sum\frac{\sin^2 n}{n^{3/2}}$

Exercise 3. **[10 points]** Compute $\displaystyle\int_0^5 \frac{x\,dx}{(x+1)(x^2+4x+3)}$.

Need to use Partial Fractions. $(x^2+4x+3) \to$ Roots $= \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm 2}{2} = -3$ & -1.

$$\frac{x}{(x+1)(x^2+4x+3)} = \frac{x}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3) + B(x+3) + C(x+1)^2}{(x+1)^2(x+3)}.$$

Let $x=-1$: $-1 = 0 + 2B + 0$. $\boxed{B = -\frac{1}{2}}$

$x=-3$: $-3 = 0 + C(-2)^2 = 4C \to \boxed{C = -\frac{3}{4}}$

For A: Take $x=0$

$0 = 3A + 3B + C = 3A - \frac{3}{2} - \frac{3}{4}$

$0 = A - \frac{1}{2} - \frac{1}{4} = A - \frac{3}{4}$ so $\boxed{A = \frac{3}{4}}$

$$\int_0^5 \frac{x\,dx}{(x+1)^2(x+3)} = \int_0^5 \frac{+\frac{3}{4}}{(x+1)} + \frac{(-\frac{1}{2})}{(x+1)^2} - \frac{3}{4}\frac{1}{x+3}\,dx$$

$$= \frac{3}{4}\ln(x+1) + \frac{1}{2}\frac{1}{(x+1)} - \frac{3}{4}\ln(x+3)\Big|_0^5$$

$$= \frac{3}{4}\ln(6) + \frac{1}{12} - \frac{3}{4}\ln 8 - \left(\frac{3}{4}0 + \frac{1}{2} - \frac{3}{4}\ln 3\right)$$

$$= \ln\left(\frac{18}{8}\right)^{3/4}\frac{1}{(3)^{3/4}} + \frac{1}{12} - \frac{1}{2} = \ln\left(\frac{18}{8}\right)^{3/4} - \frac{5}{12}$$

$$= \boxed{\ln\left(\frac{9}{4}\right)^{3/4} - \frac{5}{12}}$$

Exercise 4. **[10 points]** Compute $\int_0^\infty \underbrace{(x-1)^2}_{v}\underbrace{e^{-x}dx}_{du}$. $= \lim_{t\to\infty} \int_0^t (x-1)^2 e^{-x} dx$

Use Parts. $u = \int e^{-x} dx = -e^{-x}$.

$$\int_0^t (x-1)^2 e^{-x} dx = -(x-1)^2 e^{-x}\Big|_0^t - \int_0^t \underbrace{-e^{-x}}\, 2(x-1)\, dx$$

$$= -(x-1)^2 e^{-x}\Big|_0^t + 2\int_0^t \underbrace{e^{-x}}_{du}(x-1)\, dx$$

$$= -(x-1)^2 e^{-x}\Big|_0^t + 2\Big(-e^{-x}(x-1)\Big|_0^t \underbrace{- \int_0^t (-e^{-x})\, 1\, dx}_{+\int_0^t e^{-x} dx = -e^{-x}\Big|_0^t}\Big)$$

$$= -e^{-x}\Big(+(x-1)^2 + 2(x-1) + 2\Big|_0^t\Big)$$

$$= -e^{-x}\Big(x^2 - 2x + 1 + 2x - 2) + 2\Big|_0^t\Big) = -e^{-t}\big(+t^2 + 1\big) - \Big(\underbrace{-e^0(0+1)}_{-1.}\Big)$$

$$= -e^{-t}(t^2+1) + 1.$$

$$\lim_{t\to\infty} \underset{\downarrow 0}{-e^{-t}}\big(\underset{\downarrow \infty.}{t^2+1}\big) = -\lim_{t\to\infty} \frac{t^2+1}{e^t} = -\lim_{t\to\infty} \frac{2t}{e^t} = \dots$$

so l'Hop. $\frac{\infty}{\infty}$ $\quad 2t \to -\infty$, $e^t \to \infty$

$$= \lim_{t\to\infty} \frac{2}{e^t} = \boxed{0}$$

$e^t \to \infty$

$$\int_0^\infty (x-1)^2 e^{-x} dx = 0 + 1 = \boxed{1}.$$

Exercise 5. **[10 points]** Compute $F(t) := \int_0^t \frac{1-\sin(2x)}{\cos^2(2x)}dx$ for $0 \le t < \pi/4$.

$$\int_0^t \frac{1}{\cos^2(2x)} - \frac{\sin 2x}{\cos^2 2x}\,dx = \frac{1}{2}\int_0^{2t} \frac{1}{\cos^2 u} - \frac{\sin u}{\cos^2 u}\,du = \frac{1}{2}\int_0^{2t} \sec^2 u\,du + \frac{1}{2}\int_0^{2t} \frac{\sin u}{\cos^2 u}\,du$$

$u = 2x$
$du = 2\,dx$
$x = 0 \rightsquigarrow u = 0$
$x = t \rightsquigarrow u = 2t.$

$$= \frac{1}{2}\tan u \Big|_0^{2t} + \frac{1}{2}\frac{-1}{\cos u}\Big|_0^{2t} = \frac{1}{2}(\tan 2t - 0) - \frac{1}{2}\left(\frac{1}{\cos 2t} - \frac{1}{1}\right)$$

$$= \frac{\tan 2t}{2} - \frac{1}{2\cos 2t} + \frac{1}{2}$$

$$= \frac{\sin 2t - 1}{2\cos 2t} + \frac{1}{2}$$

Note: when $t = \frac{\pi}{4}$ $\cos 2t = \cos\frac{\pi}{2} = 0$. so we have an issue.
But $\sin\frac{\pi}{2} - 1 = 0$, so we have a $\frac{0}{0}$ indet.

Bonus problem: **[5 points]** Decide if F can be extended continuously to $t = \pi/4$, and if so, find $F(\pi/4)$.

We can extend F continuously if and only if $\lim_{t\to\pi/4^-} \frac{\sin 2t - 1}{2\cos 2t}$ exists!

$\tilde{t} = 2t$ $\quad \lim_{\tilde{t}\to\frac{\pi}{2}^-} \frac{\sin\tilde{t} - 1}{2\cos\tilde{t}} \underset{\text{L'Hop.}}{=} \lim_{\tilde{t}\to\frac{\pi}{2}^-} \frac{\cos\tilde{t}}{-2\sin\tilde{t}} = \frac{0}{-2} = \boxed{0}$

F can be extended to $\frac{\pi}{4}$ in a continuous fashion &.

$$F\left(\frac{\pi}{4}\right) = \lim_{t\to\pi/4^-} F(t) = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}.$$

For Grader's use only:

1		2		3	4	5	TOTAL
a	b	a	b				