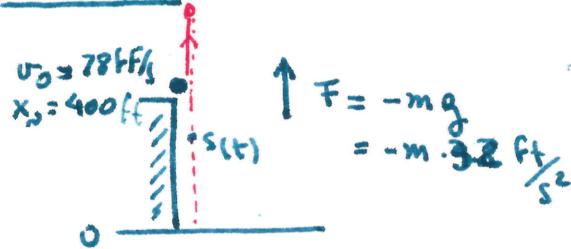


Solutions Practice Midterm 2

Problem 1:



$$m a(t) = -mg \quad \text{so} \quad a(t) = -32$$

Integrate to get velocity

$$v(x) = \int_0^x a(t) dt = -32x + C$$

$$\text{By initial conditions: } v(0) = v_0 = 78 = 0 + C \quad \text{so } C = 78$$

$$v(t) = -32t + 78 \quad \text{Integrate again to get } s(t)$$

$$s(t) = \int_0^t -32x + 78 dx = -\frac{32}{2} t^2 + 78t + C$$

$$\text{By initial conditions } 400 = s(0) = 0 + 0 + C \quad \Rightarrow C = 400.$$

Conclusion: $s(t) = -16t^2 + 78t + 400$

The ball hits the ground when $s(t) = 0$, so we need to solve

$$-16t^2 + 78t + 400 = 2(-8t^2 + 39t + 200) = 0$$

$$t = \frac{-39 \pm \sqrt{(39)^2 + 4 \cdot 8 \cdot 200}}{2(-8)} = \frac{39 \pm \sqrt{(39)^2 + 6400}}{16} = \frac{39 \pm \sqrt{7921}}{16}$$

$$= \frac{39 \pm 89}{16} \quad \begin{cases} \frac{128}{16} = 8 \\ \frac{-50}{16} < 0 \text{ no, discard} \end{cases}$$

Answer = In 8 seconds it hits the ground

Problem 2: (1) $\frac{\tan(3x)}{4x} = \frac{\sin(3x)}{\cos(3x)} \cdot \frac{1}{4x} = \frac{\sin 3x}{3x} \cdot \frac{3}{4 \underbrace{\cos(3x)}_{\downarrow x \rightarrow 0}}$

$$\text{So } \lim_{x \rightarrow 0} \frac{\tan(3x)}{4x} = \boxed{\frac{3}{4}}$$

(2) Know $\left(1 + \frac{1}{h}\right)^h \xrightarrow[h \rightarrow \infty]{} e$ We need to make $2n^2$ appear in the exponent!

$$\left(1 + \frac{1}{2n^2}\right)^{2n} = \left(\left(1 + \frac{1}{(2n)^2}\right)^{2n^2}\right)^{\frac{2n}{2n^2}} = \left(\left(1 + \frac{1}{(2n)^2}\right)^{2n^2}\right)^{\frac{1}{2n}} \xrightarrow[2n \rightarrow 0]{} e^0 = 1$$

So $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2^{n^2}}\right)^{2^n} = 1$.

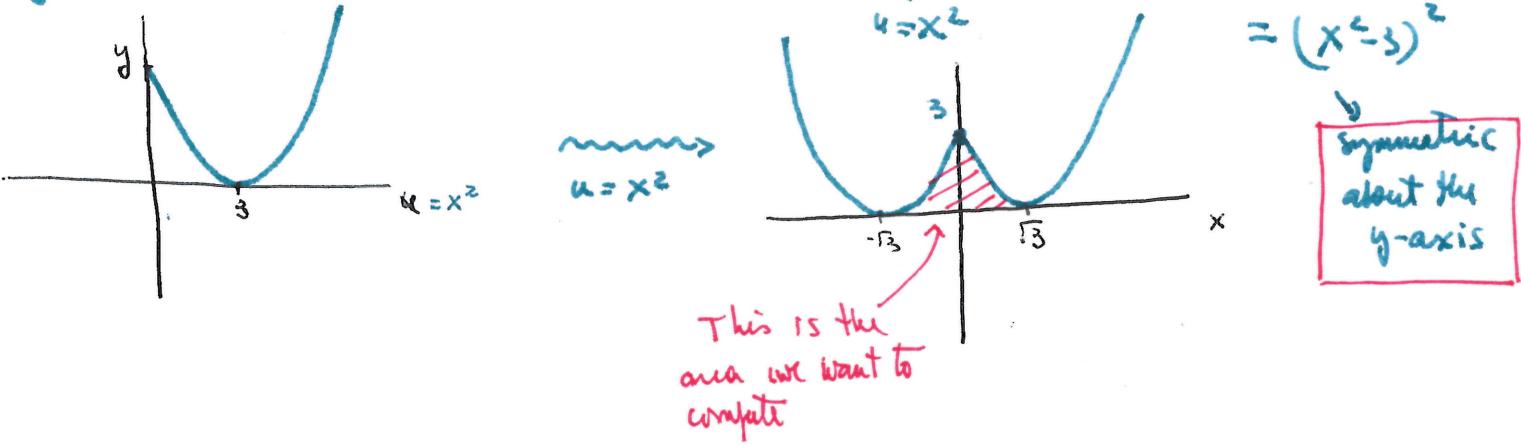
(3) Use chain rule $g(s) = \int_0^s \frac{t dt}{\sqrt{1+t^2}}$ has $g'(s) = \frac{s}{\sqrt{1+s^2}}$ by the Fund Thm of Calculus

We have $g(x^5) = \int_0^{x^5} \frac{t dt}{\sqrt{1+t^2}}$, so $\frac{d}{dx} g(x^5) = g'(x^5) \underbrace{\frac{(x^5)}{5x^4}}$

$$= \frac{x^5}{\sqrt{1+x^{10}}} \cdot 5x^4 = \boxed{\frac{5x^9}{\sqrt{1+x^{10}}}}$$

Problem 3 : We start by drawing the curve.

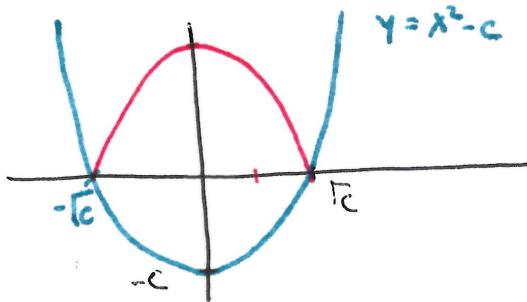
$$y = x^4 - 6x^2 + 9 = (x^2)^2 - 6(x^2) + 9 = u^2 - 6u + 9 = (u-3)^2$$



End points : $x = \pm\sqrt{3}$.

$$\begin{aligned} \text{Area} &= \int_{-\sqrt{3}}^{\sqrt{3}} y(x) dx = 2 \int_0^{\sqrt{3}} y(x) dx = 2 \int_0^{\sqrt{3}} (x^4 - 6x^2 + 9) dx \\ &= 2 \left(\frac{x^5}{5} - 2x^3 + 9x \right) \Big|_0^{\sqrt{3}} = 2 \left(\frac{(\sqrt{3})^5}{5} - 2(\sqrt{3})^3 + 9\sqrt{3} \right) \\ &= 2 \left(\frac{9\sqrt{3}}{5} - 2\sqrt{3} + 9\sqrt{3} \right) = 2\sqrt{3} \left(\frac{9}{5} - 6 + 9 \right) = 2\sqrt{3} \left(\frac{9+15}{5} \right) \\ &= \boxed{\frac{48\sqrt{3}}{5}} \end{aligned}$$

Problem 4: Start by drawing the curves:



$$\text{Curve 1 : } y = x^2 - c$$

$$\text{Curve 2 : } y = c - x^2$$

Need to determine their intersection!

$$x^2 - c = c - x^2$$

$$2x^2 = 2c$$

$$x = \pm \sqrt{c}$$

$$\text{Area} = \int_{-\sqrt{c}}^{\sqrt{c}} (\text{Curve 2} - \text{Curve 1}) dx = \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2) - (x^2 - c) dx$$

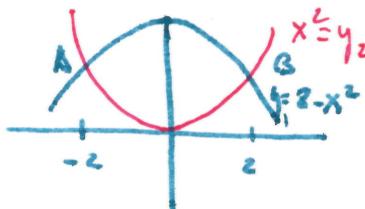
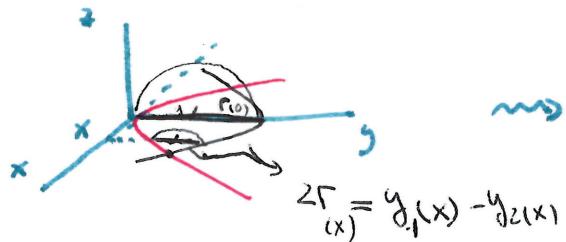
$$= \int_{\sqrt{c}}^{\sqrt{c}} 2c - 2x^2 dx = 2c x - \frac{2}{3} x^3 \Big|_{-\sqrt{c}}^{\sqrt{c}}$$

$$= 2c \underbrace{(\sqrt{c} - (-\sqrt{c}))}_{= 2\sqrt{c}} - \frac{2}{3} \underbrace{\left((\sqrt{c})^3 - (-\sqrt{c})^3 \right)}_{2c\sqrt{c}} = 4c\sqrt{c} - \frac{4}{3}c\sqrt{c} = \frac{8}{3}c\sqrt{c}$$

Want $\text{Area} = 9$ so $9 = \frac{8}{3}c\sqrt{c} \Rightarrow (\sqrt{c})^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3$ so $\sqrt{c} = \frac{3}{2}$

$$c = \frac{9}{4}$$

Problem 5: The problem tells us what the cross sections look like.



Need to find A & B

$$x^2 = 8 - x^2 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2 \quad \& \quad y = (\pm 2)^2 = 4.$$

$$\text{Vol} = \int_{-2}^2 A(x) dx \quad \& \quad A(x) = \pi(\Gamma(x))^2 = \pi ((y_1(x) - y_2(x))^2)$$

$$= \pi (8 - x^2 - x^2)^2 = \pi (2(4 - x^2))^2$$

$$= 4\pi (4 - x^2)^2 = 4\pi (16 + x^4 - 8x^2)$$

$$\text{Vol} = \int_{-2}^2 4\pi (16 + x^4 - 8x^2) dx :$$

$$\text{Vol} = 2 \int_0^2 4\pi (16 + x^4 - 8x^2) dx = 8\pi \left(16x + \frac{x^5}{5} - \frac{8}{3}x^3 \right) \Big|_0^2$$

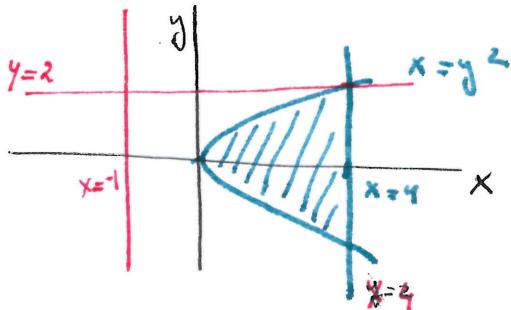
18]

$$= 8\pi \left(32 + \frac{32}{5} - \frac{8}{3} \cdot 8 \right)$$

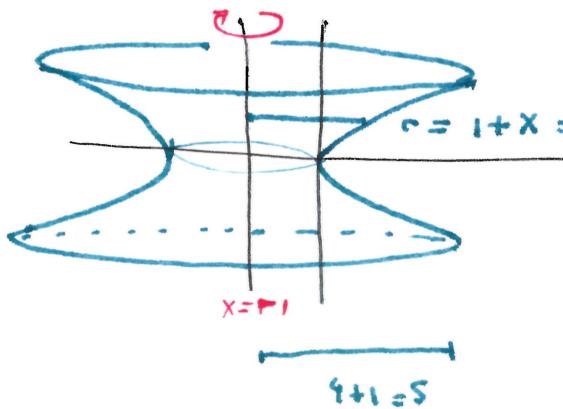
$$= 8 \cdot 32\pi \left(1 + \frac{1}{5} - \frac{4}{3} \right) = \boxed{\frac{256}{15}\pi}$$

By Symmetry

Problem 6 : As usual we start by drawing



(1) Rotate about x = -1



$$\text{Vol} = \int_{-2}^2 A(y) dy$$

$$= \int_{-2}^2 \pi (1+y^2)^2 dy$$

$$= 2 \int_0^2 \pi (1+y^2+2y^2) dy$$

Symmetry

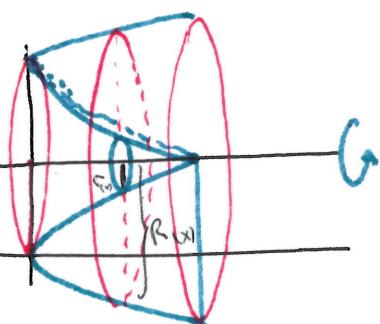
$$= 2\pi \left(y + \frac{y^5}{5} + \frac{2}{3}y^3 \right) \Big|_0^2 = 2\pi \left(2 + \frac{32}{5} + \frac{16}{3} \right) = 4\pi \left(1 + \frac{8}{15} \left(\frac{2}{5} + \frac{2}{3} \right) \right)$$

$$= 4\pi \left(1 + \frac{8}{15} \cdot 11 \right) = \frac{4\pi}{15} (15 + 88) = \boxed{\frac{4\pi}{15} (103)}$$

$$\text{Outer} = \boxed{\text{cylinder}} - \text{Vol inner} = \pi(5)^2 \cdot 4 - \frac{4\pi}{15} \cdot 103 = 4\pi (15 - \frac{103}{15}) = \frac{4\pi \cdot 272}{15} = \boxed{\frac{1088\pi}{15}}$$

(2) Rotate about y = 2

Use x-cross sections & Washer Method
(hollow part)



Alternatively : rotate outer curve & remove area of inner curve.

Washer \rightarrow



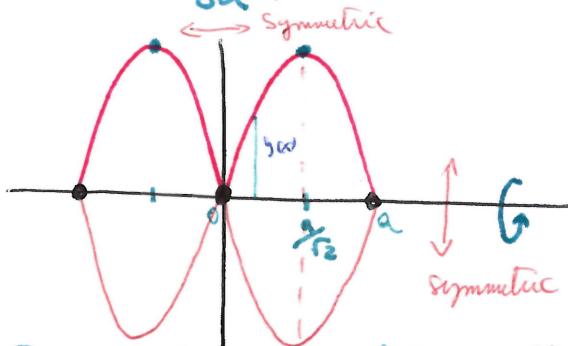
$$\begin{cases} r(x) = 2 - y_1(x) = 2 - \sqrt{x} \\ R(x) = 2 - y_2(x) = 2 + \sqrt{x} \end{cases}$$

$$\text{INNER } y_1 = \sqrt{x}, \text{ OUTER } y_2 = -\sqrt{x}$$

$$\begin{aligned}
 \text{Vol} &= \int_0^4 A(x) dx = \int_0^4 \pi(R_{(x)}^2 - r_{(x)}^2) dx \\
 &= \int_0^4 \pi((2+\sqrt{x})^2 - (2-\sqrt{x})^2) dx \\
 &= \int_0^4 \pi((\underbrace{2+\sqrt{x} + (2-\sqrt{x})}_4)(\underbrace{2+\sqrt{x} - (2-\sqrt{x})}_{2\sqrt{x}})) dx \\
 &= \int_0^4 8\pi \sqrt{x} dx = 8\pi \frac{x^{3/2}}{3} \Big|_0^4 = \frac{16\pi}{3} (4^{3/2} - 0) = \boxed{\frac{128\pi}{3}}
 \end{aligned}$$

Problem 7: $8a^2y^2 = x^2(a^2-x^2)$ If $a=0$, $0=-x^2$ is just the line $x=0$.
 gives a plane with no finite surface area. We next treat $a \neq 0$. Since a^2 appears, we can assume from now on that $a > 0$.

$$y^2 = \frac{x^2}{8a^2} (a^2-x^2).$$



By symmetry, need to compute one arch only!

We write $y = y(x)$ & need $y'(x)$ to find surface area.

Use implicit differentiation $2y y' = \frac{1}{8a^2}(x^2 a^2 - x^4)' = \frac{1}{8a^2}(2a^2 x - 4x^3)$

$$\text{so } y' = \frac{x}{8a^2} (a^2 - 2x^2)$$

$$1 + (y')^2 = 1 + \frac{x^2(a^2 - 2x^2)^2}{(8a^2)^2 y^2} \stackrel{\text{use eqn!}}{=} 1 + \frac{x^2(a^2 - 2x^2)^2}{64a^4 \frac{x^2}{8a^2} (a^2 - x^2)}$$

Take $x=0$, $x=\pm a$ to get $y=0$ values.

$$\text{For } x = \pm \frac{a}{2}, y = \frac{a^2}{4} \left(a^2 - \frac{a^2}{4} \right) = a^2 \frac{3}{2}$$

$x^2(a^2-x^2)$ is a parabola in x^2 so the vertex is at $x = \frac{a^2}{2}$ so $\boxed{x = \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}}$

$$= 1 + \frac{(a^2 - 2x^2)^2}{8a^2(a^2 - x^2)} = \frac{8a^2(a^2 - x^2) + (a^2 - 2x^2)^2}{8a^2(a^2 - x^2)}$$

$$= \frac{8a^4 - 16a^2x^2 + a^4 - 4a^2x^2 + 4x^4}{8a^2(a^2 - x^2)} = \frac{9a^4 - 12a^2x^2 + 4x^4}{8a^2(a^2 - x^2)}$$

$$= \frac{(2x^2 - 3a^2)^2}{8a^2(a^2 - x^2)}$$

$$\text{Area} = 2 \int_0^a 2\pi y \sqrt{\frac{(2x^2 - 3a^2)^2}{8a^2(a^2 - x^2)}} dx$$

Here:

$$y = \sqrt{\frac{x^2(a^2 - x^2)}{8a^2}}$$

$$\Rightarrow \frac{y(x)}{\sqrt{a^2 - x^2}} = \frac{x}{\sqrt{8a^2}}$$

$\sqrt{(2x^2 - 3a^2)^2}$ should be positive if $0 \leq x \leq a$

so we should use $-(2x^2 - 3a^2) = 3a^2 - 2x^2$

$$\text{Area} = 4\pi \int_0^a \left(\frac{x}{\sqrt{8a^2}} \right)^2 (2x^2 + 3a^2) dx$$

$$= \pi \int_{+a^2}^{+3a^2} \frac{-u}{8a^2} \frac{du}{4x} = \frac{16}{8a^2} \frac{(-u^2)}{2} \Big|_{a^2}^{3a^2}$$

$$u = -2x^2 + 3a^2 \quad u(0) = +3a^2$$

$$du = -4x dx \quad u(a) = +a^2$$

Map limits
of integration

$$= \frac{16}{16a^2} \left(9a^4 - a^4 \right)$$

$$= \boxed{\frac{1}{2}\pi a^2}$$

Problem 8:

$$v(t) = 3 \text{ ft/s} \quad \text{Duration is 10 sec.}$$

$$F(t) = m a(t) = -m g = -m \cdot 32 \text{ ft/s}^2$$

Mass is changing with time!

We need to find the position $s(t)$ given $s(t) = s(0) + \int_0^t v(u) du = \int_0^t 3 du = 3t$

$$s(t) = 3t \quad \Delta F = g m(t) = \text{original weight} - \text{rate of loss} \cdot t$$

$$= 100 - 4.5 t = 100 - 4.5 \frac{s(t)}{3} = \boxed{100 - \frac{3}{2}s(t)}$$

The force, in terms of the trajectory becomes $F(s) = 100 - \frac{3}{2}s$ (in lb)

Trajectory goes from $s(0)=0$ to $s(10) = 3 \cdot 10 = 30$.

$$dW = F ds \quad \text{gives} \quad W = \int_0^{30} F(s) ds = \int_0^{30} 100 - \frac{3}{2}s ds = 100s - \frac{3}{4}s^2$$

$$= \boxed{2325 \text{ ft-lb}}$$

(25)

Problem 9: We want the tangent line $r(x_0, y_0)$ to be

$$y = f'(x_0)(x - x_0) + y_0 \quad \text{and contain } (0, 0)$$

so $\boxed{y = f'(x_0)x \quad \& \quad \text{contain } (x_0, y_0)}$

(1) $f(x) = e^{ax}$ so $f'(x) = ae^{ax}$ & $f'(x_0) = ae^{ax_0}$

Want the line $y = ae^{ax_0}x$ to contain (x_0, e^{ax_0})

so $y_0 = e^{ax_0} = ae^{ax_0}x_0 \quad \& \quad e^{ax_0} \neq 0 \quad \Rightarrow \text{we can divide \& get} \quad 1 = ax_0.$

Answer: $\begin{cases} x_0 = \frac{1}{a} \\ y_0 = e^{\frac{1}{a}} \end{cases}$

(2) $f(x) = \ln(x)$ so $f'(x) = \frac{1}{x}$ & $f'(x_0) = \frac{1}{x_0} \Rightarrow y = \frac{1}{x_0}x$ is line.

Want to find x_0 with $\ln(x_0) = \frac{1}{x_0}x_0 = 1$.

So $\ln(x_0) = 1$ gives $x_0 = e^{\ln(x_0)} = e^1 = e$ Ans: $x_0 = e$

Problem 10: Want to rewrite $x(t) = 3\sin(2t) + 4\cos(2t)$ as $A\sin(at+b)$.

• Evaluate at various points to find A & b .

a = 2 by const.

• $t=0 \quad x(0) = 4 = A \sin(b)$.

(Diff eqn is $\ddot{x} + 4x = 0$)

• $t=\frac{\pi}{4} \quad x(\frac{\pi}{4}) = 3\sin(\frac{\pi}{2}) + 4\cos(\frac{\pi}{2}) = 3 = A\left(\sin\left(2\frac{\pi}{4} + b\right)\right) = A \sin(\frac{\pi}{2} + b)$

Now: $\sin(\frac{\pi}{2} + b) = \sin\frac{\pi}{2}\cos b + \sin b \cos\frac{\pi}{2} = \cos b$.

We get $\begin{cases} 4 = A \sin b \\ 3 = A \cos b \end{cases} \Rightarrow \tan b = \frac{4}{3} \quad \& \quad 25 = 16 + 9 = A^2 \sin^2 b + A^2 \cos^2 b = A^2 \cdot 1$

so A = ±5

Amplitude = $|A| = 5$

& Frequency = $\left(\frac{2\pi}{a}\right)^{-1} = \left(\frac{2\pi}{2}\right)^{-1} = \boxed{\pi^{-1}} = \boxed{\frac{1}{\pi}}$

Velocity: $x'(t) = 6\cos(2t) \mp 8\sin(2t)$

Need to maximize $x'(t)$.

Use expression $x(t) = A \sin(at+b)$
 $= A \sin(2t+b)$

$x'(t) = 2A \cos(2t+b)$ is max when $\cos(2t+b) = 1$ & $A = 5$
 $\cos(2t+b) = -1$ & $A = -5$

In both cases : $x'(t)$ has max value = 10 (twice times the amplitude)