

Lecture I: §2.1 & §2.2 Tangents

Textbook: Calculus with Analytic Geometry (G.F. Simmons), 2nd Edition.

§2.1. What is Calculus? The Problem of Tangents

(a) Why?

- Many of the ideas of calculus ("to compute") go back to Archimedes & the Greeks.
- Modern perspective & its formalization is credited to
 - Newton (1642-1727)
 - Leibniz (1646-1716)

Since then, it has become part of the basic language of science, in particular the description of continuous motions & processes.

- Applications (Newton): motions of planets + gravity.
- Vital to Physics & Engineering (since 17th century)
- Nowadays: Basic vocabulary of sciences as diverse as:
 - Biology: Hodgkin-Huxley equation describing the action potential across neurons in the brain.
 - Economics: Black-Scholes eqn. modeling option pricing in financial markets.
- Why formalize & axiomatize? Allows us to describe diverse phenomena by concentrating on the underlying structure - free from the constraining vocabulary of the specific field / example / phenomena we are studying. Gain flexibility by means of abstraction.

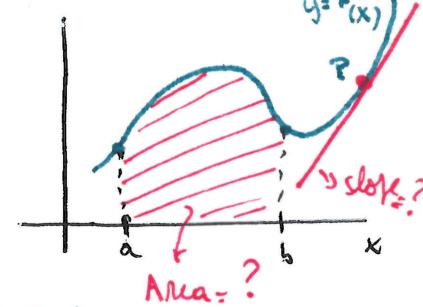
↳ 2 Fundamental Questions:

- Q1: Find the rate at which a variable quantity is changing.
- Q2: Describe a varying quantity when its rate of change is known.

Simmons recasts these as 2 geometric problems

P1: Finding tangent lines to curves (\leftrightarrow calculate their slope m)
 $y = mx + b$)

P2: Find the area under a graph.

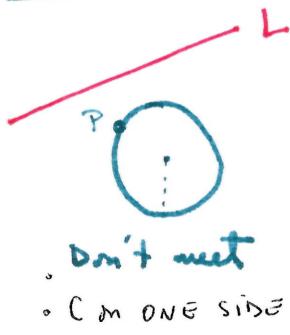


{ P1 \leftrightarrow Differential Calculus (Part I of this course)
P2 \leftrightarrow Integral Calculus (Part II —————)
(Part III: Infinite series)

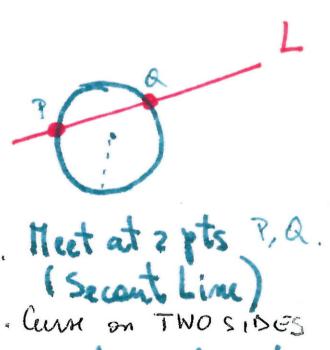
• Connection between P1 & P2 = Fundamental Thm of Calculus (§ C.6)

(c) What is a tangent line?

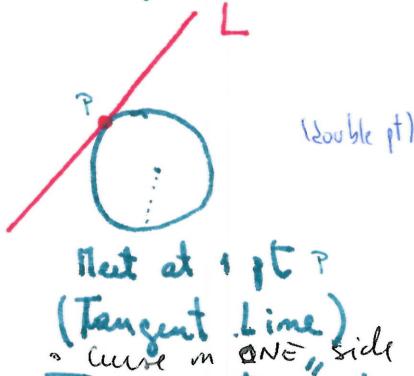
• Idea 1: Look at lines relative to curves (e.g. circles)



• Don't meet
• On ONE SIDE



• Meet at 2 pts P, Q.
(Secant Line)
• Curve on TWO SIDES

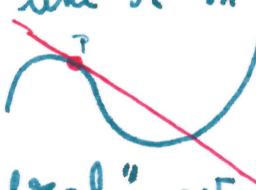


• Meet at 1 pt P
(Tangent Line)
• Curve on ONE side

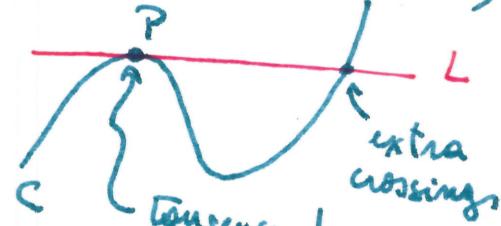
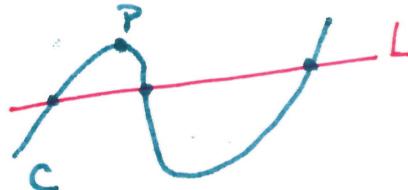
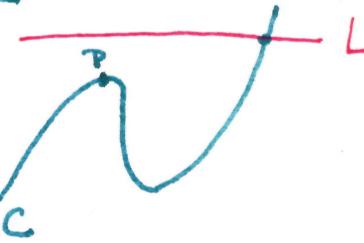
• Notion of "counting the number of intersection points" fails for general curves.

• Idea 2: Curve on "one side" of the line or on both & meeting point.

This also fails in general



Fix: Combine these 2 ideas in a "local" situation (around a point P in the curve C)



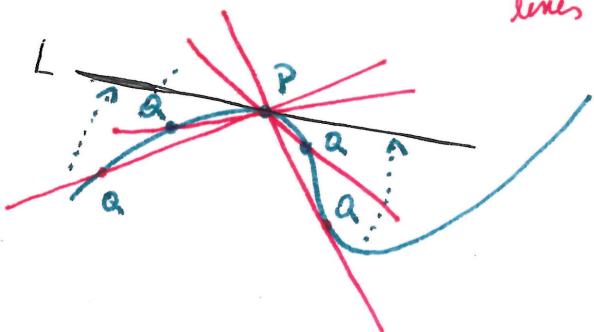
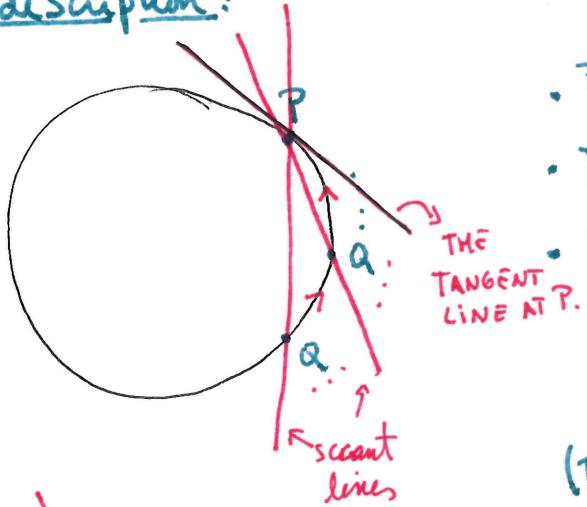
tangency!

extra crossings

Natural Question: How to think about this more formally?

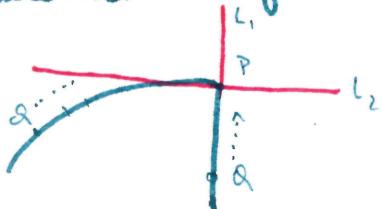
A (Fermat v1630) "Tangent lines are LIMITS of secant lines"

Geometric description:



⚠: Tangents don't always exist!

Eg



Q: Which one is the tangent?

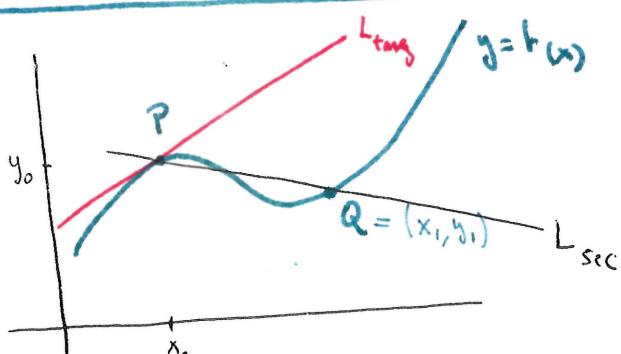
A: None!

Fermat's idea

- Fix P & a second point Q
 - Draw the secant line through P & Q
 - Move Q towards P, the secant line PQ approaches (the limiting) tangent line to the curve C at P.
- (Provided this line exists!)

Note: Almost all of calculus involves limiting processes of one type or other

§ 2.2 How to calculate the slope of the tangent?



Equation of a line through $P = (x_0, y_0)$

$$y = m x + b$$

m slope b constant

$$y = \boxed{m(x - x_0) + y_0}$$

Recall: m should be obtained by a limiting process.

⇒ We need coordinates!

Eqn of secant line $PQ = L_{sec}$ $y - y_0 = m_{sec} (x - x_0)$

where

$$\boxed{\frac{y_1 - y_0}{x_1 - x_0} = m_{sec}}$$