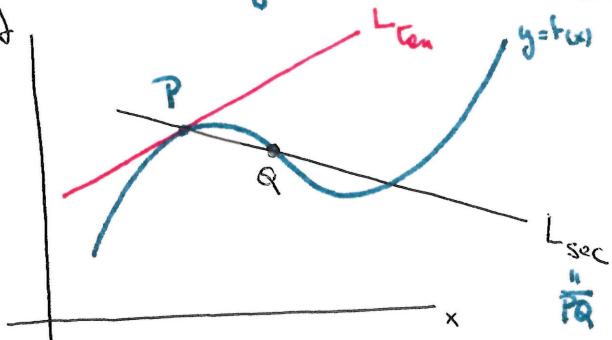


6

Lecture II : § 2.2 (cont) Slopes of tangents
 § 2.3 : Definition of derivative.

Recall: "Tangent lines are limits of secant lines".



$P = (x_0, y_0)$ fixed
 $Q = (x_1, y_1)$ approaches P

$$L_{\tan} : y = m_{\tan}(x - x_0) + y_0$$

$$L_{\sec} : y = m_{\sec}(x - x_0) + y_0$$

where $m_{\sec} = \frac{y_1 - y_0}{x_1 - x_0}$

§ 2.2 Slopes of tangents

Since $L_{\tan} = \lim_{Q \rightarrow P} L_{\sec}$
 ↓ ↓
 slope m_{\tan} slope m_{\sec}

, then $m_{\tan} = \lim_{Q \rightarrow P} m_{\sec}$

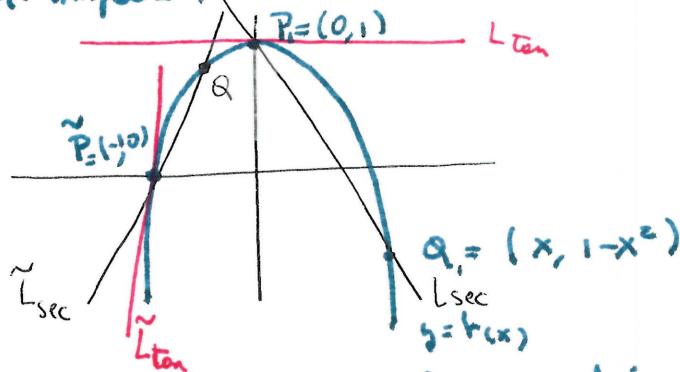
so $m_{\tan} = \lim_{\substack{x_1 \rightarrow x_0 \\ y_1 \rightarrow y_0}} \frac{y_1 - y_0}{x_1 - x_0}$

Important fact: We always have $Q \neq P$ so if the curve is the graph of a function $y = f(x)$, we have $x_1 \neq x_0$, $y_0 = f(x_0)$ & $y_1 = f(x_1)$

$$m_{\tan} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} (= f'(x_0))$$

• Some numerical examples:

$$y = 1 - x^2$$



Ex 1: $P_1 = (0, 1)$
 $(x_0 = 0)$

$$m_{\tan} = \lim_{x \rightarrow 0} \frac{(1-x^2)-1}{x-0} = \lim_{x \rightarrow 0} \frac{-x^2}{x} = \lim_{x \rightarrow 0} -x = 0$$

\therefore horizontal!

Ex 2: $\tilde{P} = (-1, 0)$

$$m_{\tan} = \lim_{x \rightarrow -1} \frac{(1-x^2)-0}{x-(-1)} = \lim_{x \rightarrow -1} \frac{1-x^2}{x+1} = \lim_{x \rightarrow -1} \frac{(1-x)(1+x)}{x+1}$$

$$\boxed{x+1 \neq 0} \rightarrow = \lim_{x \rightarrow -1} 1-x = 1-(-1) = 2$$

- Conclusion:
- Geometry tells us what the tangent should be
 - Analysis allows us to formally compute it
(choose the point P , compute m_{\tan} as a limit to get L_{\tan})

Formal procedure = Method of increments

- Think of $P = (x_0, y_0)$ as being fixed
- Moving the point $Q = (x_1, y_1)$ towards $P \Rightarrow$ interpret this as

$$x_1 = x_0 + \Delta x \quad , \quad y_1 = y_0 + \Delta y$$

↓ increment in x ↑ increment in y

where Δx & Δy are small increments (positive or negative!) that we let go to zero.

Now: $m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$, so $m_{tan} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

- Special case: $y = f(x)$, so $y_0 = f(x_0)$, $y_1 = f(x_1)$

Then substitute in $\Delta y = y_1 - y_0 = f(x_1) - f(x_0)$
 $= f(x_0 + \Delta x) - f(x_0)$

Conclusion: $m_{sec} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

$$m_{tan} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Examples above (revisited) $y = 1 - x^2$, $P = (0, 1)$, $\tilde{P} = (-1, 0)$

Get $m_{sec} = \frac{1 - (x_0 + \Delta x)^2 - (1 - x_0^2)}{\Delta x} = \frac{-(x_0^2 + (\Delta x)^2 + 2x_0 \Delta x) + x_0^2}{\Delta x}$
 $= -\frac{\Delta x(\Delta x + 2x_0)}{\Delta x}$

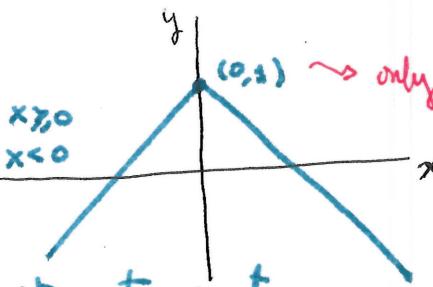
Ex 1 $m_{sec} = -\frac{\Delta x(\Delta x + 2 \cdot 0)}{\Delta x} = -\frac{(\Delta x)^2}{\Delta x} \stackrel{\Delta x \neq 0}{=} -\Delta x \xrightarrow[\Delta x \rightarrow 0]{} 0$

Ex 2 $m_{sec} = -\frac{\Delta x(\Delta x + 2(-1))}{\Delta x} \stackrel{\Delta x \neq 0}{=} 2 - \Delta x \xrightarrow[\Delta x \rightarrow 0]{} 2$

As expected, the answers are the same (but we used different techniques)
 computational

- Example with no Tangent

$$y = f(x) = 1 - |x| = \begin{cases} 1-x & x \geq 0 \\ 1+x & x < 0 \end{cases}$$



All pts except (0,1) admit a tangent

Increment method gives $y = x + 1$ for $P = (x_0, y_0)$ with $x_0 < 0$

$$y = 1 - x \text{ for } x_0 > 0.$$

[Also: curve is piecewise linear with corner braces = (0,1)]

$$\text{At } P = (0,1) \quad m_{\text{Tan}} = \lim_{\Delta x \rightarrow 0} \frac{1 - |\Delta x| - 1}{\Delta x} = \lim_{\Delta x \rightarrow 0} -\frac{|\Delta x|}{\Delta x} = \begin{cases} -1 & \text{if } \Delta x > 0 \\ 1 & \text{if } \Delta x < 0 \end{cases}$$

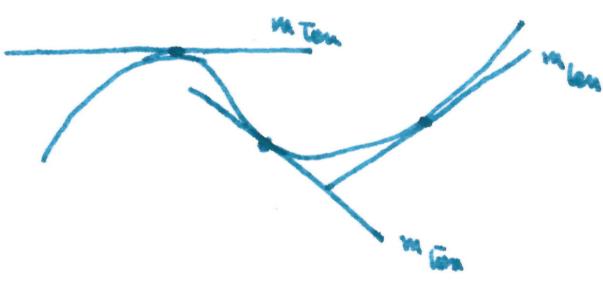
so the limit does NOT exist! (sidewise limits do not agree!)

§ 2.3 Derivative of a function:

- (1) What is a function?
- Def.: Given a set of numbers D , a function $f: D \rightarrow \mathbb{R}$ (defined in D)
 is a formula / rule / law of correspondence that assigns a single real number y to each number x in D . We write $y = f(x)$, or

$$x \mapsto y$$

[single value = "critical test"]
- D = domain of f
- The values y assigned by f are called the range of f .
- x = independent variable, y = dependent variable (b/c it depends on x)
- We have been computing slopes of tangent lines at points



Thus, m_{Tan} defines a function

This seems to assign to each P on the graph a line L_{Tan} , or a number, namely the slope m_{Tan}

Using coordinates, we have $y = f(x), \Rightarrow$
 $\boxed{x \mapsto \text{slope of } L_{\text{Tan}} \text{ at } P}$

$$P = (x, f(x))$$

Our assignment rule ("formula") for "then" is

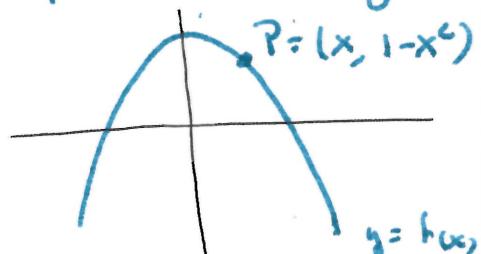
$$x \longmapsto \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} =: f'(x) \quad (\text{if the limit exists}).$$

This function is the derivative of $f(x)$ ("it is derived from the func $y = f(x)$ we write it as $f'(x)$).

Note: Since the limit need not exist, the domain of the function f' might be smaller than the domain of f .

For each concrete f , we can find a formula for f' , using the method of increments for an arbitrary point $P = (x, f(x))$.

Eg 1: $f(x) = 1 - x^2$



$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(1 - (x + \Delta x)^2) - (1 - x^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 - (x^2 + 2x\Delta x + (\Delta x)^2) - 1 + x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-2x - \Delta x) = -2x. \end{aligned}$$

Compare with numerical examples:

$$\left. \begin{array}{ll} \text{At } x=0 & f'(0) = -2 \cdot 0 = -0 = 0 \checkmark \\ x=-1 & f'(-1) = -2(-1) = 2 \checkmark \end{array} \right\} \text{slopes we computed before.}$$

Eg 2 $f(x) = 1 - |x| \Rightarrow f'(x) = \begin{cases} 1 & x < 0 \\ \text{undefined} & x=0 \\ -1 & x > 0 \end{cases}$

