

Lecture III § 2.3 (cont) Definition of derivative

§ 2.5. The concept of a limit. Two trigon. limits

§ 2.3 Recall: Given $f: D \rightarrow \mathbb{R}$ a function with $D \subseteq \mathbb{R}$, we define f'

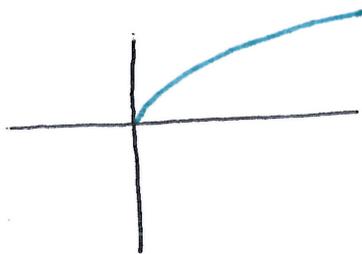
$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (\text{whenever the limit for } x \in D \text{ exists})$$

Eg: $f(x) = x^3 \implies f'(x) = 3x^2$

Try these as (exercise)

$f(x) = \frac{1}{x}$ for $D = \{x > 0\} \implies f'(x) = -\frac{1}{x^2}$

$f(x) = \sqrt{x}$ for $D = \{x \geq 0\} \implies f'(x) = \frac{1}{2\sqrt{x}}$ for $\{x > 0\}$



$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

Trick: multiply by $1 = \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$

So $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x} (\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ whenever $x > 0$

Notation:

Leibniz: $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

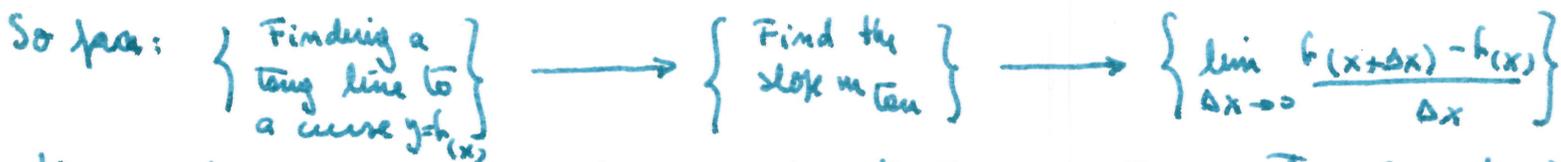
(IDEA: infinitesimal change in y / infinitesimal change in x)

$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \left(\frac{d}{dx} \right) f$

↳ operation performed on the function f

• "Correct notation can clarify concepts!"

§ 2.5 What is a limit?



We interchange an undefined but intuitive geometric notion (tangent line) with a similarly intuitive yet undefined notion (limit)

Q: What does $\lim_{x \rightarrow a} g(x) = L$ mean? Here, g is a function defined around a , but not necessarily at a .

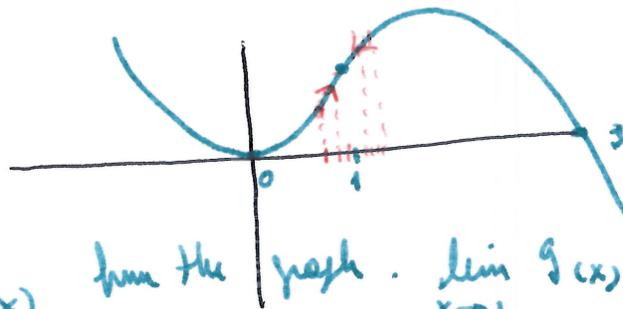
(*) As x approaches a , the value of $g(x)$ approaches L

(**) We can make $g(x)$ as close as we want to L by taking x close enough to a .

Note: We are not claiming anything about the value $g(a)$ (and we shouldn't care about it!)

The way to understand this notion depends on the way we understand the function $g(x)$.

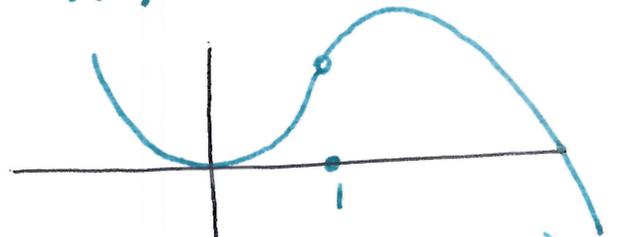
• Eg 1 $g(x) = 3x^2 - x^3$



We can guess $\lim_{x \rightarrow 1} g(x)$ from the graph. $\lim_{x \rightarrow 1} g(x) = \lim_{\Delta x \rightarrow 0} g(1 + \Delta x) = g(1) = 2$

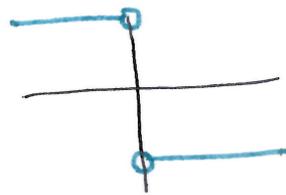
Similarly for other values: $\lim_{x \rightarrow a} g(x) = g(a)$.

• Eg 2: $g(x) = \begin{cases} 3x^2 - x^3 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1 \end{cases}$

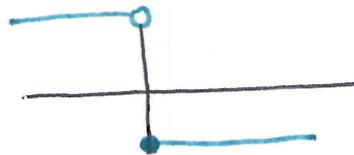


Still $\lim_{x \rightarrow 1} g(x) = 2$ (away from $x=1$ both pictures agree).

• Eg 3: $f(x) = \begin{cases} -x & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}$



$\lim_{x \rightarrow 1} f(x) = 1$ but $\lim_{x \rightarrow 0} f(x)$ does NOT exist



• Eg 4: $f(x) = \begin{cases} -1 & \text{for } x \geq 0 \\ 1 & \text{for } x < 0 \end{cases}$

$\lim_{x \rightarrow 1} f(x) = -1$, $\lim_{x \rightarrow 0} f(x)$ does not exist

(Same picture as Eg 3 away from $x=0$)

Q What can we say if we don't know the graph of g ?

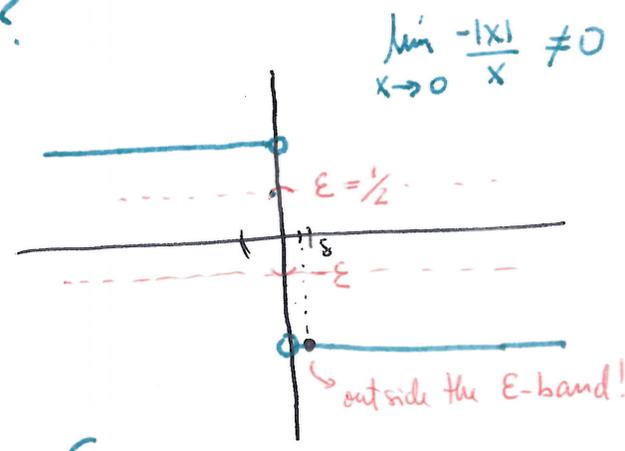
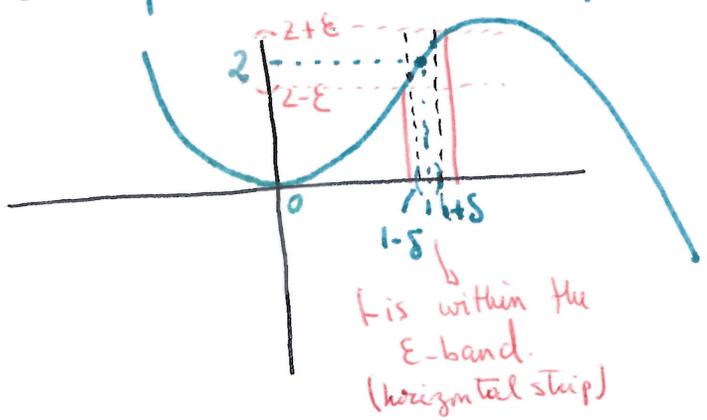
We make $(**)$ more precise! Game of choices

[YOUR CHOICE]

• Pick how close to L you want to be (say $|g(x) - L| < 10^{-8}$)
← challenge me to determine how close to a must x be so
if say $0 < |x - a| < 10^{-26}$ then for all these x we
can ENSURE $|g(x) - L| < 10^{-8}$.
[MY PICK]

Def We say $\lim_{x \rightarrow a} f(x) = L$ if for every $\epsilon > 0$ (your choice)
, there is a $\delta > 0$ (my choice) such that if $0 < |x - a| < \delta$,
then $|f(x) - L| < \epsilon$. (ie I always win the game of choice)
[rules out $x \neq a$]

• How to fit this with the graphs/pictures?



↳ any $\delta \neq 0$ $f(\delta/2) = \pm 1$.
($|\delta/2 - 0| < \delta$) & $|f(\delta/2) - 0| > \frac{1}{2}$

Note: Even if we have a formula for f , finding δ given ϵ can be challenging (because it involves "inverting" a formula)

Eg: $f(x) = 1 - x^2$. Prove $\lim_{x \rightarrow 0} f(x) = 1$

Big E-delta:
if $|x - 0| < \delta$ then $|f(x) - L| = |1 - x^2 - 1| = |-x^2| = x^2 < \epsilon$
How to find δ given ϵ ?
We know $x^2 < \delta^2$, so pick $\delta^2 = \epsilon$, i.e. $0 < \delta = \sqrt{\epsilon}$