

## Lecture IV: §2.5 Two trigonometric limits

### §2.4 Rates of change & velocity

#### §2.5 Two trigonometric limits

- Using the  $\epsilon/\delta$  method (for a formal proof) we discover the following principles for how limits behave:

Prop: Assume  $\lim_{x \rightarrow a} f(x) = L$  &  $\lim_{x \rightarrow a} g(x) = M$ . Then,

$$(1) \lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M$$

$$(2) \lim_{x \rightarrow a} [f(x)g(x)] = LM$$

$$(3) \text{For any real number } c, \lim_{x \rightarrow a} [cf(x)] = cL$$

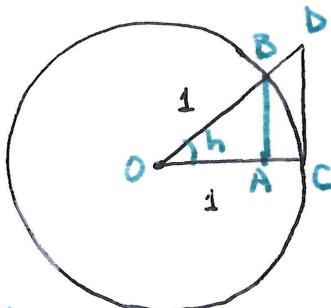
$$(4) \text{If } M \neq 0, \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proofs: Lecture V (Appendix A2)

### Two trigonometric limits:

E.g.:  $\lim_{h \rightarrow 0} \frac{\sin h}{h}$  .  $\sin(0)=0$  so we have a % indeterminate form.  
Q: Do things cancel out, like  $\frac{x^c}{x}$ ?

- $\sin(h)$  is related to trigonometry in the unit circle:



( $h > 0$ )

$$\frac{AB}{1} = \sin(h) \quad \& \quad \frac{OA}{1} = \cos(h)$$

$$\frac{CD}{1} = \tan(h)$$

a sector  $OBC$  of the circle

We have an arc  $= BC$ ,  $\nabla$  & two triangles  $\triangle OAB$  &  $\triangle OCD$

Area of  $\triangle OBC$  = Area circle  $\cdot \frac{h}{2\pi} = \frac{h}{2}$   
= fraction of the circle

Area of  $\triangle OAB$  =  $\frac{1}{2} \text{base} \cdot \text{height} = \frac{1}{2} \cos h \cdot \sin h$

• Area of  $\triangle ABC$  =  $\frac{1}{2}$  base · height =  $\frac{1}{2} \cdot 1 \cdot \tanh h = \frac{\sinh h}{2 \cosh h}$

(inner triangle) Area  $\triangle OAB$  Area sector " " Area  $\triangle OBC$  (outer triangle)

So  $\frac{1}{2} \cosh h \sinh h \leq \frac{h}{2} \leq \frac{1}{2} \frac{\sinh h}{\cosh h}$

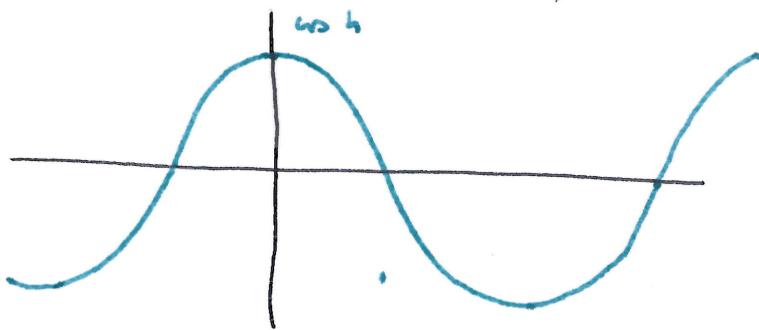
If  $h > 0$ : divide by  $\frac{h}{2}$  everywhere

$$\cosh \frac{\sinh h}{h} \leq 1 \leq \frac{1}{\cosh h} \frac{\sinh h}{h} \quad \text{so } \cosh \leq \frac{\sinh h}{h} \leq \frac{1}{\cosh h}$$

$(\cosh > 0 \text{ for } h > 0 \text{ near } 0)$

If  $h < 0$   $\cosh \frac{\sinh h}{h} \geq 1 \geq \frac{1}{\cosh h} \frac{\sinh h}{h} \quad \text{so } \cosh \leq \frac{\sinh h}{h} \leq \frac{1}{\cosh h}$

$(\cosh < 0 \text{ for } h < 0 \text{ near } 0)$



We get

$$\cosh \leq \frac{\sinh h}{h} \leq \frac{1}{\cosh h}$$

We know  $\lim_{h \rightarrow 0} \cosh = 1 = \lim_{h \rightarrow 0} \frac{1}{\cosh h}$

So  $\frac{\sinh h}{h}$  is squeezed between 1 & 1 in the limit, so

$$\boxed{\lim_{h \rightarrow 0} \frac{\sinh h}{h} = 1}$$

Application:  $\lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} = ?$  (also % type) both pieces have a limit!

$$\frac{\cosh h - 1}{h} \cdot \frac{\cosh h + 1}{\cosh h + 1} = \frac{\cosh^2 h - 1}{h(\cosh h + 1)} = \frac{-\sinh^2 h}{h(\cosh h + 1)} = \frac{+\sinh h}{h} \cdot \frac{-\sinh h}{\cosh h + 1}$$

$$\text{So } \lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} = \lim_{h \rightarrow 0} \frac{\sinh h}{h} \cdot \lim_{h \rightarrow 0} \frac{-\sinh h}{\cosh h + 1} = 1 \cdot \left(-\frac{1}{2}\right) \cdot 0 = \boxed{0}$$

## §2.4: Velocity & rates of change

If we think of  $y = f(x)$  as indicating a relationship between physical quantities, then:

$\frac{\Delta y}{\Delta x}$  becomes the average rate of change

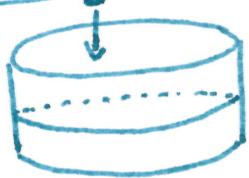


$f' = \frac{dy}{dx}$  becomes the (instantaneous) rate of change

[ Problem of tangent slopes becomes the Problem of rate of change.]

Applications: The independent variable  $x=t$  is TIME.

Ex ① Filling a water tank



$V(t)$  = Volume at time  $t$

$\frac{dV}{dt}$  = rate at which the tank is filled

Also: the height  $h(t)$  is changing so

$\frac{dh}{dt}$  = rate at which the height changes.

Note:  $V$  &  $h$  are related  $V(t) = h(t) \cdot \text{Area of the base}$ .

(E.g.:  $h(t) = t$ , then  $\frac{dV}{dt} = \frac{d}{dt}(t \cdot \text{Area}) = \text{Area}$  of the base)

Ex ② Rock falling of a cliff:

rock

$$s(t) = 16t^2 \text{ (ft)}$$
$$s'(t) = 32t \text{ (ft/s)} \text{ (prove it by the definition)}$$
$$a(t) = 32 \text{ (ft/s}^2\text{)} \text{ (From GRAVITY!)}$$

• Speed:  $|v(t)| \rightarrow$  no direction.

$s(t)$  = position at time  $t$

•  $\frac{\Delta s}{\Delta t} = s'(t) = v(t)$  velocity (has a direction)  $> 0$  or  $< 0$

•  $\frac{\Delta v}{\Delta t} = v'(t) = a(t)$  acceleration

$v(t)$  = rate of change of the position

$a(t) = \frac{\Delta v}{\Delta t}$  velocity

(14)

Application 2 : The independent variable can be something else, for example  $x = \#$  cars produced.

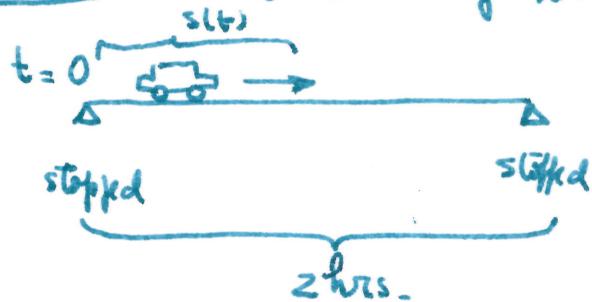
$C(x) = \text{cost of producing } x \text{ cars.}$

$$\frac{dC}{dx} = \text{marginal cost}$$

(Econ. or Business term for rate of change)

or "cost per unit" (we don't make fractional cars  
1 car = instantaneous)

Application 3 : Car driving straight down the road.



• Odometer = measures distance travelled  
• speedometer = — speed (forward velocity)

Watching odometer gives  $s(t) = \text{distance at time } t$

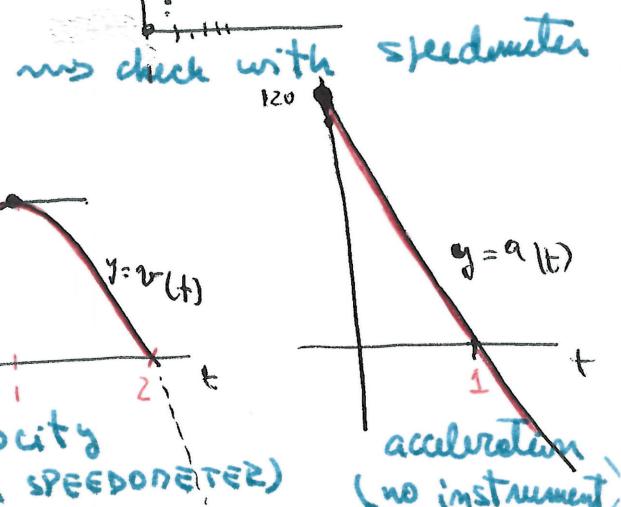
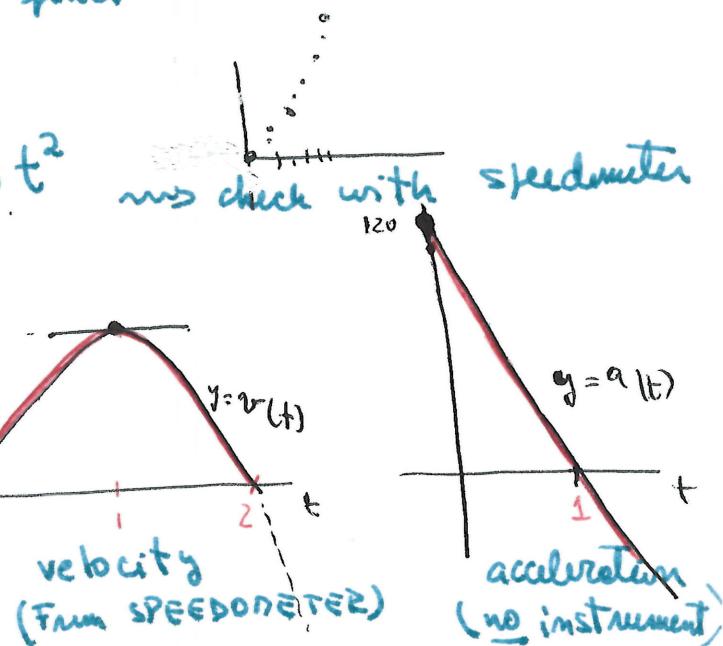
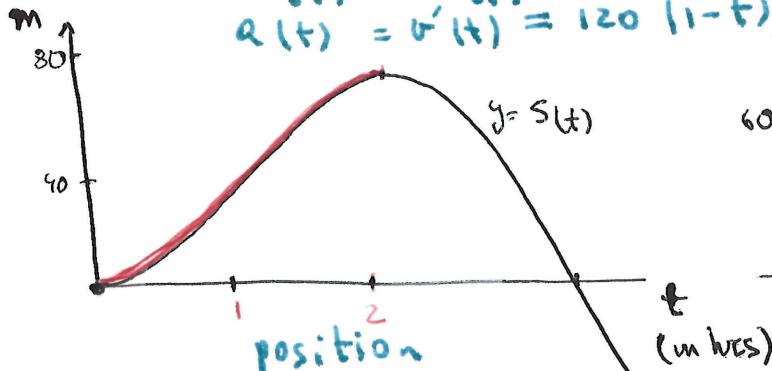
— speedometer —  $v(t) = s'(t) = \text{velocity at time } t$

By empirical measure, one might find or estimate that

$$s(t) = 60t^2 - 20t^3$$

$$\text{So } v(t) = s'(t) = 120t - 60t^2$$

$$a(t) = v'(t) = 120(1-t)$$



We stop measuring after  $t = 2$  hours.

• Each graph gives a "different" picture of our travels