

Lecture V : Appendix A.2 : Theorems about limit

Def.: Given a function f defined around a point a (not necessarily at a) we say $\lim_{x \rightarrow a} f(x) = L$ (in \mathbb{R}) if for every $\epsilon > 0$, we can find δ (dependent on ϵ & a) such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$

Eg: $f(x) = \begin{cases} 5x+4 & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$ Pick $a=0$. Guess: $\lim_{x \rightarrow 0} f(x) = 4$

WANT $|5x+4 - 4| = |5x| = 5|x| < \epsilon$ if $|x| < \delta$.

Take $\delta = \frac{\epsilon}{5}$. Then $|5x| < 5 \cdot \delta = 5 \cdot \frac{\epsilon}{5} = \epsilon$ ✓

Eg 2 $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ (Lecture IV), $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \approx 2.71828$ (Chapter 8)

- Natural question 1: Can we approach 2 different limits L, L' ? four
- " " 2: How do limits behave with respect to the operations $+, -, \cdot, /$ in \mathbb{R} ? What about inequalities?

Thm 1: If $\lim_{x \rightarrow a} f(x) = L$ & $\lim_{x \rightarrow a} f(x) = L'$, then $L=L'$.

Proof: If $L \neq L'$, then $\tilde{\epsilon} = |L-L'| > 0$. Pick $\epsilon = \frac{\tilde{\epsilon}}{2} > 0$

By def.: we can find $\delta_1 > 0$ so that $0 < |x-a| < \delta_1$ ensures $|f(x) - L| < \epsilon$

Pick $\delta = \min\{\delta_1, \delta_2\}$ $\delta_2 > 0$ $\quad \quad \quad 0 < |x-a| < \delta_2 \quad \quad |f(x) - L'| < \epsilon$

Now: $\epsilon = |L-L'| = |\underbrace{f(x)-L'}_{(*)} + \underbrace{L-f(x)}| \leq |\underbrace{f(x)-L'}_{(*)}| + |\underbrace{f(x)-L}| < 2\epsilon$

We conclude $2\epsilon < 2\epsilon$, but this is not possible! (because $|x-a| < \delta \leq \delta_1$) (because $|x-a| < \delta \leq \delta_2$)

So what went wrong? A: Our original assumption $L \neq L'$ leads to a contradiction, so it must be false. We conclude that $\boxed{L = L'}$ ■

Note: This is an example of a proof by contradiction. $\frac{L-E}{L+E} < \frac{\epsilon}{2\epsilon}$

Why (*)? $|c+d| \leq |c| + |d|$ for any pair of real numbers c, d .

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Some easy limits:

Thm 2: $\lim_{x \rightarrow a} x = a$ & $\lim_{x \rightarrow a} C = C$ for any constant C

Proof. Want $|x-a| < \epsilon$ if $|x-a| < \delta$. Pick $\delta = \epsilon$
 • " $0 = |C-C| < \epsilon$ " if $|x-a| < \delta$ Pick ANY δ , $\delta > 0$
 example $\delta = 1$. \blacksquare

Thm 3: If $\lim_{x \rightarrow a} f(x) = L$ & $\lim_{x \rightarrow a} g(x) = M$, then.

$$(1) \lim_{x \rightarrow a} (f(x) \pm g(x)) = L \pm M$$

$$(2) \lim_{x \rightarrow a} f(x)g(x) = L \cdot M$$

Proof (1) Pick any $\epsilon > 0$. By the definition of L , $\exists \delta_1 > 0$ s.t. $\tilde{\epsilon} = \epsilon/2$
 we can find $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x)-L| < \frac{\epsilon}{2}$
 $\underline{\delta_1 > 0} \quad \underline{0 < |x-a| < \delta_1} \quad |g(x)-M| < \frac{\epsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$ & assume $0 < |x-a| < \delta$. We want to
show that $|f(x)+g(x)-(L+M)| < \epsilon$. (*)

$$|f(x)+g(x)-(L+M)| = |f(x)-L + g(x)-M| \leq \underbrace{|f(x)-L|}_{< \frac{\epsilon}{2}} + \underbrace{|g(x)-M|}_{< \frac{\epsilon}{2}} \quad \begin{matrix} \text{(because } \delta < \delta_1\text{)} \\ \text{(because } \delta < \delta_2\text{)} \end{matrix}$$

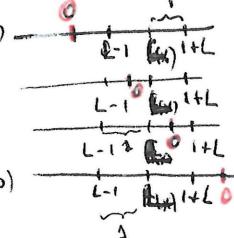
We have confirmed (*).

The claim for $f(x) \cdot g(x)$ has exactly the same proof.

$$\begin{aligned} (2) |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x)-M) + (f(x)-L)M| \leq |f(x)(g(x)-M)| + |(f(x)-L)M| \\ &= |f(x)| |g(x)-M| + |f(x)-L| |M| \leq |f(x)| |g(x)-M| + |f(x)-L| (|M| + 1) \end{aligned}$$

Q: Can we approximate each summand? Pick a suitable $\tilde{\epsilon}$ for each term:

- For $f(x)$: find $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x)-L| < 1 (\tilde{\epsilon})$
 so $-1+L < f(x) < 1+L$ & so $|f(x)| < 1+|L|$.
- For $g(x)$: find $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then
 $|g(x)-M| < \frac{\epsilon}{2}(1+|L|)$



• For $f(x)$, find $\delta_3 > 0$ so that if $0 < |x-a| < \delta_3$, then

$$|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$$

Pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $|f(x)g(x) - LM| \leq |f(x)| |g(x) - M| + |f(x) - L| (|M| + 1) < (1+|L|) \frac{\epsilon}{2(1+|L|)} + \frac{\epsilon}{2(1+|M|)} (1+|M|) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

so $|f(x)g(x) - LM| < \epsilon$ as we wanted.

Consequence: For any $f(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$ in $\mathbb{R}[x]$,

$$\lim_{x \rightarrow a} f(x) = c_n a^n + c_{n-1} a^{n-1} + \dots + c_0 = f(a).$$

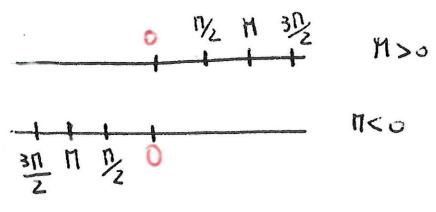
Theorem 4: If $\lim_{x \rightarrow a} g(x) = M \neq 0$, then $\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$.

Proof: Write $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x)M} \right| = |M - g(x)| \frac{1}{|M||g(x)|}$

Since $M \neq 0$, we know $|M| > 0$. Take $\tilde{\epsilon} = \frac{|M|}{2} > 0$ & pick $\delta_1 > 0$ so that $|g(x) - M| < \tilde{\epsilon}$,

$$M - \frac{|M|}{2} = -\tilde{\epsilon} + M < g(x) < \tilde{\epsilon} + M = \frac{|M|}{2} + M$$

Thus $\frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$, so $\left| \frac{1}{g(x)} \right| < \frac{2}{|M|}$



Pick $\delta_2 > 0$ so that $|g(x) - M| < \frac{\epsilon |M|^2}{2}$ whenever $0 < |x-a| < \delta_2$

Take $\delta = \min\{\delta_1, \delta_2\}$ & assume $0 < |x-a| < \delta$. Then:

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = |M - g(x)| \frac{1}{|M||g(x)|} < \frac{\epsilon |M|^2}{2} \cdot \frac{1}{|M|} \cdot \frac{2}{|M|} = \epsilon$$

Consequence: If $\lim_{x \rightarrow a} g(x) = M \neq 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

$$\lim_{x \rightarrow a} f(x) = L$$

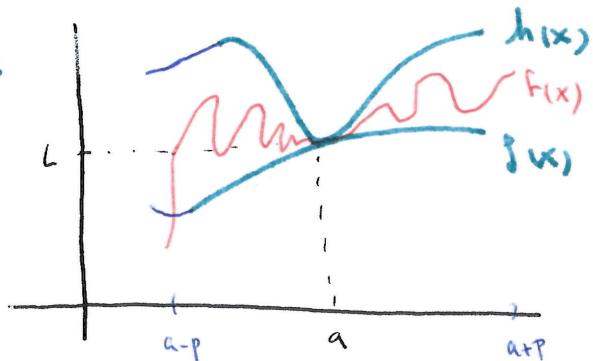
(Write $f(x)/g(x) = f(x) \cdot \frac{1}{g(x)}$ & continue Thus 3 & 4).

Squeeze Them: Assume $g(x) \leq f(x) \leq h(x)$ in a neighborhood of a
 (if $0 < |x-a| < p$)
 \Rightarrow since $p > 0$ (from δ)
 $\lim_{x \rightarrow a} g(x)$ exists & $\lim_{x \rightarrow a} h(x)$ exists &

If $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then the $\lim_{x \rightarrow a} f(x)$ exists &
 its value is also L .

Note: We used this in Lecture V to show $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Proof:



Pick $\delta_1 > 0$ so that if
 $0 < |x-a| < \delta_1$, then $|h(x)-L| < \varepsilon$

Pick $\delta_2 > 0$ so that if
 $0 < |x-a| < \delta_2$ then $|g(x)-L| < \varepsilon$

But $|h(x)-L| < \varepsilon$ means $L-\varepsilon < h(x) < L+\varepsilon$
 $|g(x)-L| < \varepsilon$ — $L-\varepsilon < g(x) < L+\varepsilon$

Pick $\delta = \min\{\delta_1, \delta_2\} > 0$ & assume $0 < |x-a| < \delta$. Then

$$L-\varepsilon \leq \underbrace{g(x)}_{\delta \leq \delta_2} \leq \underbrace{f(x)}_{\delta \leq \delta} \leq \underbrace{h(x)}_{\delta \leq \delta_1} \leq L+\varepsilon, \text{ so } |f(x)-L| < \varepsilon$$

(look at the ends!)