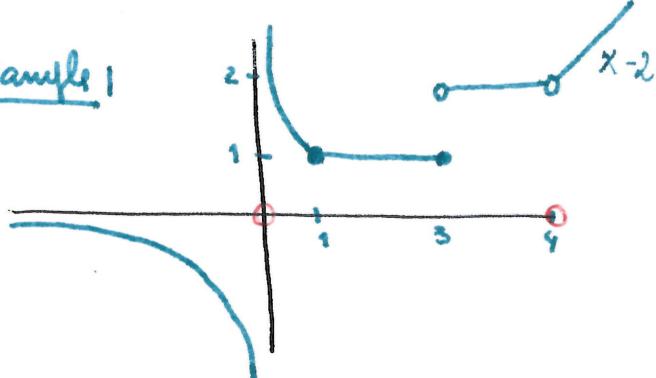


Lecture VI : § 2.6 Continuous functions & the mean value theorem

Example 1



$$f(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \frac{1}{x} & 0 < x < 1 \\ 1 & 1 \leq x \leq 3 \\ 2 & 3 < x < 4 \\ x-2 & x > 4 \end{cases}$$

$$\text{Domain } D = \mathbb{R} \setminus \{0, 4\}$$

• f is continuous at all pts in D except $x=3$: $\lim_{x \rightarrow 3^+} f(x) = 2 \neq f(3)$

• We can extend f to $x=4$ in a continuous way by declaring $f(4) := \lim_{x \rightarrow 4} f(x) = 2$

• We can't extend f to $x=0$ in a continuous way since $\lim_{x \rightarrow 0^+} f(x) = \infty$

Example 2 (Lecture V) Polynomials are continuous functions on \mathbb{R} .

Def: A function f defined in a neighborhood of a is continuous at a if:

(1) f is defined at a

(2) $\lim_{x \rightarrow a} f(x)$ exists and $= f(a)$.

Def: f is continuous on an interval $[a, b]$ if it is continuous on all pts of the interval.

Useful notation: $\lim_{x \rightarrow a} f(x) = f(a)$ is the same as $\lim_{x \rightarrow a} (f(x) - f(a)) = 0$

We rewrite this by increments $\underbrace{\lim_{\Delta x \rightarrow 0} f(a + \Delta x) - f(a)}_{=\Delta f} = 0$

Proposition: If f is differentiable at a (meaning $f'(a)$ exists), then f MUST be continuous at a .

Proof: $\lim_{\Delta x \rightarrow 0} \Delta f = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} \cdot \Delta x = \boxed{\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}} \underset{= f'(a)}{\underset{\substack{\text{Product Rule} \\ \lim_{\Delta x \rightarrow 0} \Delta x = 0}}{\underset{\Delta x \rightarrow 0}{\rightarrow}}} \underset{\Rightarrow 0}{\underset{\Delta x \rightarrow 0}{\rightarrow}} \boxed{0}$

Remark: The other implication fails: for example $f(x) = |x|$ is continuous at $x=0$ but not differentiable at $x=0$.

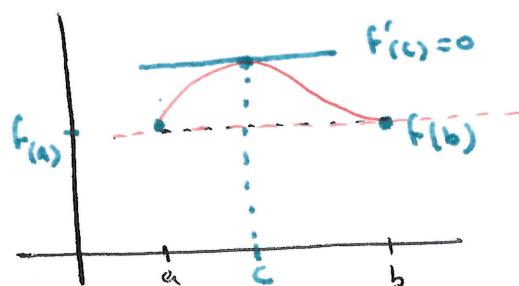
3.1 The Mean Value Theorem

MVT: Consider a function $f: [a, b] \rightarrow \mathbb{R}$ where:

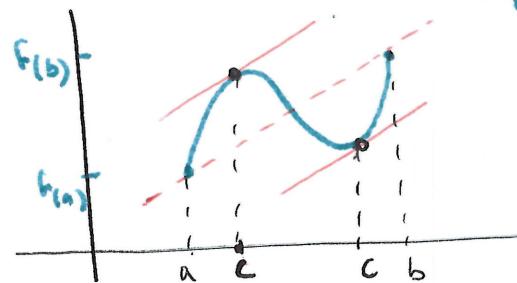
(1) f is continuous on $[a, b]$

(2) f is differentiable on (a, b) [excluding $x=a$ & $x=b$]

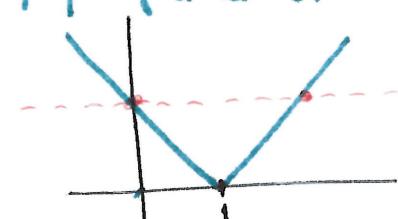
Then we can find some c in (a, b) where $f'(c) = \frac{f(b) - f(a)}{b-a}$



$f(a) = f(b)$ \Rightarrow no value for c



2 values for c



$$f(x) = |x - 1|$$

$$a=0, b=2$$

no wing Tangent
(don't have $f'(1)$)

Proof: See Appendix A4 (Lecture XII).

Important consequences: Assume f continuous on $[a, b]$ & differentiable on (a, b)

Consequence 1: If $f'(x) > 0$ on (a, b) then f is strictly increasing on $[a, b]$

Proof: Take $a < a' < b' < b$. Want to show $f(a) < f(b')$.

By MVT, since f is cont on $[a', b']$ & differentiable on (a', b') we can find c in (a', b') with $f'(c) = \frac{f(b') - f(a')}{b' - a'} > 0$

so $0 < f(b') - f(a')$, equivalently $f(a') < f(b')$, as desired \blacksquare

Consequence 2: If $f'(x) < 0$ on (a, b) then f is strictly decreasing on (a, b) .

Consequence 3: If $f'(x) \equiv 0$ on (a, b) , then f is constant on (a, b) .

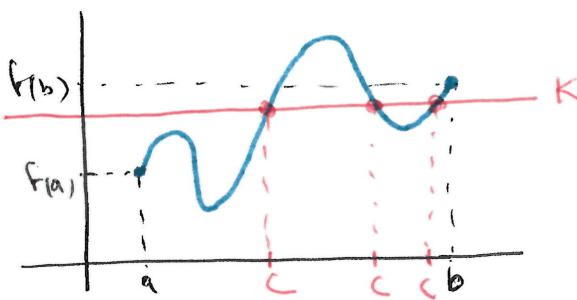
Proof: Take $a < a' < b' < b$. By MVT, can find c in (a', b') with $f'(c) = \frac{f(b') - f(a')}{b' - a'} = 0$ so $0 = f(b') - f(a')$ or

$$f(b') = f(a') \text{ for any } a', b' \quad \blacksquare$$

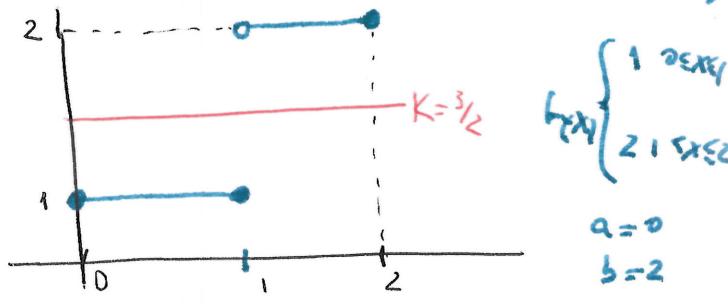
Remark: The sign of f' allows us to predict the growth of f .

3.2 The Intermediate Value Thm

IVT: If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then every K between $f(a)$ & $f(b)$ is attained, meaning we can find c in $[a, b]$ with $K = f(c)$



The graph crosses the horiz line $y = K$ three times!



The theorem fails for $K = 3/2$

Reason: f is not continuous (not continuous on $x=1$)

Special case: If $f(a) > 0$ & $f(b) < 0$ (or vice versa), we can find c in $[a, b]$ with $f(c) = 0$

$$\text{Eg: } f(t) = 3t^2 - t^3 + 1$$

↓ dominant term

$$\lim_{t \rightarrow \infty} f(t) = -\infty$$

$$\lim_{t \rightarrow -\infty} f(t) = +\infty$$

so it has a real root!

t	$f(t)$
0	$1 > 0$
3	$1 > 0$
4	$48 - 64 + 1 = -15 < 0$

} we have a root in $(3, 4)$

We can refine our search by computing the sign of $f(3.5)$, ...
(Bisect the intervals so that the size goes to 0 to approximate the location of the root.)