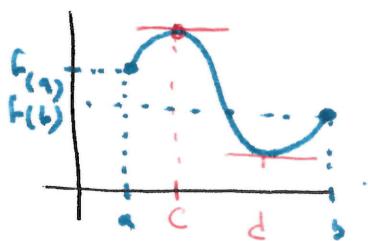


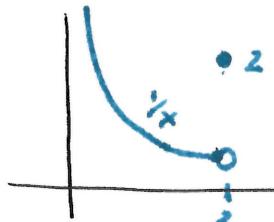
Recall: $f: (c, d) \rightarrow \mathbb{R}$ is continuous at a if (1) f is defined at a ,
 (2) $\lim_{x \rightarrow a} f(x) = f(a)$.
 (so $c < a < d$)

§ 1. The Extreme Value Thm

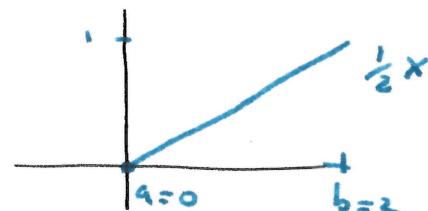
EVT If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains both a maximum & a minimum value in $[a, b]$. ("extreme values")



$f(c)$ is MAX
 $f(d)$ is MIN
 $(f$ differentiable at (a, b))
 $\& f'(c) = f'(d) = 0$)



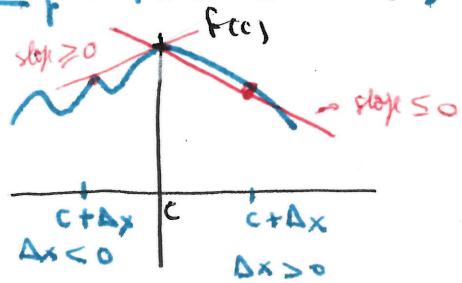
$\lim_{x \rightarrow a^-} f(x) = +\infty$
 no max attained
 no min attained
 f not defined at $x=a$
 — cont. at $x=b$



$f(a)$ is MIN
 $f(b)$ is MAX
 f differentiable BUT
 $f'(a) \neq 0, f'(b) \neq 0$

Consequence: If c in (a, b) is an extreme value for f , f is continuous on $[a, b]$ & f is differentiable at c , then $f'(c) = 0$.
 (horiz tangent line at $(c, f(c))$)

Proof: Assume $f(c)$ is MAXIMUM (proof for $f(c)$ is MIN is analogous)



$$\text{We know } f'(c) = \lim_{\Delta x \rightarrow 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Now, we look at the signs of both the numerators & the denominator:

If $\Delta x > 0$: $\left. \begin{array}{l} f(c+\Delta x) - f(c) \leq 0 \\ \Delta x > 0 \end{array} \right\}$ so slope of the secant is ≤ 0
 Thus, the limit $f'(c)$ is ≤ 0

If $\Delta x < 0$: $\left. \begin{array}{l} f(c+\Delta x) - f(c) \leq 0 \\ \Delta x < 0 \end{array} \right\}$ so slope of the secant is ≥ 0
 Thus, the limit $f'(c)$ is ≥ 0

But if $f'(c) \leq 0$ & $f'(c) \geq 0$ is only possible if $f'(c) = 0$ □

§2 Derivatives of Polynomials

TODAY: Basic rules of derivation & how to apply them.

Start with 2 "easy" derivatives:

Prop: (1) $\frac{d}{dx} c = 0$ (derivative of constant functions is 0)

(2) $\frac{d}{dx} x^n = nx^{n-1}$ for n positive integer.

Proof: (1) By definition $\frac{d}{dx} c = \lim_{\Delta x \rightarrow 0} \frac{c(x+\Delta x) - c(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$

$$(2) \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = ?$$

Need to rewrite the numerator: $(a+b)^n = \underbrace{(a+b)(a+b)\dots(a+b)}$

We use the distributive formula to get Binomial Thm: ^{n times}

$$\begin{aligned} (a+b)^n &= a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots + \frac{n(n-1)\dots(n-k+1)}{1\cdot 2 \dots k} a^{n-k} b^k \\ &\quad + \dots + ab + b^n \\ &= a^n + na^{n-1}b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{k} a^{n-k} b^k + \dots + \binom{n}{n-1} b^{n-1} + b^n \end{aligned}$$

$$\text{Eg: } (a+b)^2 = a^2 + 2ab + b^2, \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \quad \text{"n choose k"}$$

$$\text{In our case: } (x+\Delta x)^n = x^n + nx\Delta x + \dots + \binom{n}{k} x^{n-k} (\Delta x)^k + \dots + nx(\Delta x)^{n-1} + (\Delta x)^n$$

$$\text{So } \frac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1} \cancel{\frac{\Delta x}{\Delta x}} + \dots + \binom{n}{k} x^{n-k} \cancel{\frac{(\Delta x)^k}{\Delta x}} + \dots + \cancel{\frac{(\Delta x)^n}{\Delta x}}.$$

$$\text{So } \binom{n}{k} x^{n-k} \cancel{\frac{(\Delta x)^{k-1}}{\Delta x}} \xrightarrow[\Delta x \rightarrow 0]{} 0 \quad \text{for } 2 \leq k \leq n, \quad \text{so only the term } n x^{n-1} \text{ survives.}$$

$$\text{We conclude: } \frac{d}{dx} x^n = nx^{n-1}.$$

To use Prop to understand derivatives of polynomial functions

$$P(x) = a_n x^n + \dots + a_1 x + a_0$$

$a_0, a_1, \dots, a_n \in \mathbb{R}$,
n positive integer or $n=0$

We need 2 more derivative rules:

Theorem 1: If c is a constant & $f(x)$ is differentiable at x , then the function $g(x) = cf(x)$ is also differentiable & $g'(x) = cf'(x)$.

$$\begin{aligned} \text{Proof: } g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} c \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) = c \frac{df}{dx}(x) = cf'(x) \quad \blacksquare \end{aligned}$$

Product rule
to limits

Consequence: $\frac{d}{dx}(cx^n) = cnx^{n-1}$ (derivative of a constant times a monomial x^n)

Theorem 2: If $f(x)$ & $g(x)$ are both differentiable at x , then so is $h(x) = f(x) + g(x)$ & $h'(x) = f'(x) + g'(x)$.

$$\begin{aligned} \text{Proof: } \lim_{\Delta x \rightarrow 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{(f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = f'(x) + g'(x) \quad \blacksquare \end{aligned}$$

rearrange
& distribute Δx

$\downarrow \Delta x \rightarrow 0$ $\downarrow \Delta x \rightarrow 0$ \downarrow
 $f'(x)$ $g'(x)$ Sum rule
for limits

Consequence of Theorems 1 & 2: If $f = a_n x^n + \dots + a_1 x + a_0$ is a polynomial of degree $n \geq 0$, then $\frac{df}{dx} = a_n n x^{n-1} + a_{n-1} (n-1) x^{n-2} + \dots + a_2 2 x + a_1$ (degree = $n-1$)

Q: How to interpret Theorems 1 & 2? "Differentiation is a linear operator on the space of functions".

Linear: sends sums to sums.

• sends multiplication by scalars (constants) to mult. by scalars

$$(f+g) \mapsto f' + g' = (f+g)'$$

To mult. by scalars

$$c \cdot f(x) \mapsto c f'(x) = (cf)(x)$$

These properties & spaces with sums & mult by scalars are the subject of Linear Algebra.

Exercise 1: Assume $s(t) = 12 - 6t + 3t^2$ is the position at time t of an object moving in a straight line. Compute the velocity, the acceleration.

Soln: $v(t) = s'(t) = -6 + 6t = 6(t-1)$

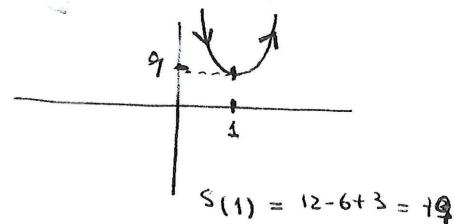
$a(t) = v'(t) = 6 \cdot 1 = 6$

. At what time does the object change direction?

$v(t) = 0 \Rightarrow t=1$ If $t < 1$ $v(t) < 0$ ($s(t)$ is decreasing)

If $t > 1$ $v(t) > 0$ ($s(t)$ is increasing)

So the object changes direction at $t=1$.



Exercise 2 Find the points on the curve $y = 4x^3 + 6x^2 - 24x + 10$ at which the tangent is horizontal.

Soln $f(x) = 4x^3 + 6x^2 - 24x + 10 \Rightarrow f'(x) = 12x^2 + 12x - 24$
 $= 12(x^2 + x - 2)$

or use roots: $\frac{-1 \pm \sqrt{1^2 - 4 \cdot 1(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$ so $x = 1$ or $x = -2$.

Horizontal if $f'(x) = 0$, so $x = 1$ or $x = -2$.

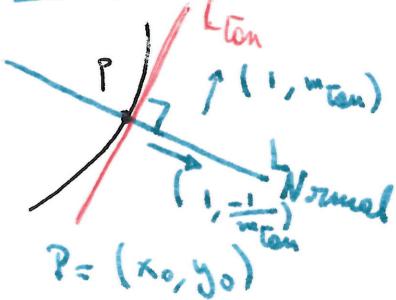
Exercise: Find the vertex of the parabola $y = x^2 - 4x + 5$

Soln: At the vertex, the tangent line is horizontal, hence slope = 0

$f'(x) = 2x - 4 = 0 \Rightarrow x = 2$.

The y-value is $f(2) = 4 - 8 + 5 = 1$ so Vertex = (2, 1).

Normal to a curve at P = line through P perpendicular to the tangent line.



$y = m_{\text{tan}}(x - x_0) + y_0$. If L_{tan} is vertical ($x = x_0$) L_{normal} is horiz ($y = y_0$)

$y = m_{\text{norm}}(x - x_0) + y_0$. If L_{tan} is horiz ($y = y_0$) L_{normal} is vertical ($x = x_0$)

$m_{\text{norm}} = -\frac{1}{m_{\text{tan}}}$ \Rightarrow (Hint for Ex 22) then L_{normal} is vertical ($x = x_0$)