

## Lecture VIII: § 3.2 The Product & Quotient Rules

Last time: Rules for differentiation under addition, powers (with integer & positive exponents) and multiplication by constants.

### § 1 Product Rule:

Theorem 1: Given  $f(x), g(x)$  two functions differentiable at  $x$ , then the product  $h(x) = f(x)g(x)$  is also differentiable at  $x$  &

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\frac{d}{dx}g(x).$$

Proof:  $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} = ?$  We add & subtract  $f(x)g(x+\Delta x)$

& regroup :

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(f(x+\Delta x) - f(x))g(x+\Delta x) + f(x)(g(x+\Delta x) - g(x))}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x+\Delta x) + f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

sum rule for limits

$$\frac{df}{dx} \quad (\text{f diff})$$

$\downarrow \Delta x \rightarrow 0$   
because  $g$  is cont. at  $x$   
(diff  $\Rightarrow$  cont.)

$\downarrow \Delta x \rightarrow 0$   
 $f(x)$   
(nothing more!)

$\downarrow \Delta x \rightarrow 0$   
 $\frac{dg}{dx}$   $(g$  diff)

$$\stackrel{\uparrow}{=} \frac{df}{dx} g(x) + f(x) \frac{dg}{dx}.$$

Example 1: Verify that this agrees with the Power Rule (Lecture VI)

$$4x^3 = \frac{d}{dx}(x^4) = \frac{d}{dx}(x^3 \cdot x) = \frac{d}{dx}x^3 \cdot x + x^3 \frac{dx}{dx} = 3x^2 \cdot x + x^3 = 4x^3 \quad \checkmark$$

Power rule

Distrub

$$\underline{\text{Example 2}} \quad f = (2x-5)(x^3-4x+8) \stackrel{\uparrow}{=} 2x^4 - 5x^3 - 8x^2 + 36x - 40$$

$$\text{Use Prod Rule: } f' = 2(x^3-4x+8) + (2x-5)(3x^2-4) = \stackrel{\uparrow}{2x^3-8x+16} + 6x^3 - 15x^2 - 16x + 20.$$

$$\text{Example 3} \quad (x^{n+m})' = (n+m)x^{n+m-1}$$

Using Product Rule:  $(x^n)' x^m + x^n (x^m)' = n x^{n-1} x^m + x^n m x^{m-1}$   
 $= n x^{n+m-1} + m x^{n+m-1}$   
 $= (n+m) x^{n+m-1}$

## § 2 Quotient Rule

Q: Given  $f(x)$  &  $g(x)$ , when can we define  $h(x) = \frac{f(x)}{g(x)}$ ?

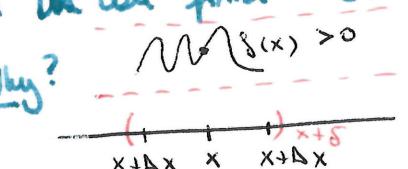
A:  $h$  is defined whenever  $g(x) \neq 0$ .

Domain of  $h = \{x : f \text{ & } g \text{ are both defined at } x \text{ & } g(x) \neq 0\}$

How to differentiate  $h(x)$ ? Need to know how to differentiate  $\frac{1}{g(x)}$   
& use the product rule  $h' = f'(x) \cdot \frac{1}{g(x)} + f\left(\frac{1}{g(x)}\right)'$ .

Thm 2: Assume  $g(x) \neq 0$  & that  $g(x)$  is differentiable at  $x$ . Then  
 $h(x) = \frac{1}{g(x)}$  is defined in a neighborhood of  $x$  & it is differentiable  
at  $x$  with  $h'(x) = -\frac{g'(x)}{g(x)^2}$ .

Proof: If  $g$  is differentiable at  $x$ , then  $g$  is continuous at  $x$ .

Since  $g(x) \neq 0$ , then we can find  $\delta > 0$  with  $0 < |\Delta x| < \delta$ , then  
 $g(x + \Delta x) \neq 0$ . [Why? 

If  $g(x) > 0$ , take  $\varepsilon = g(x)/2$   
 $g(x) - \varepsilon < g(x + \Delta x) < g(x) + \varepsilon$   
 $0 < \frac{\varepsilon}{2} < \frac{g(x)}{2}$

So we can write down  $h(x + \Delta x)$ .

$$\frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{\frac{1}{g(x + \Delta x)} - \frac{1}{g(x)}}{\Delta x} = \frac{\frac{g(x) - g(x + \Delta x)}{g(x + \Delta x) g(x)}}{\Delta x}$$

$$= \frac{1}{g(x + \Delta x)} - \frac{1}{g(x)} \frac{g(x) - g(x + \Delta x)}{\Delta x} \xrightarrow[\Delta x \rightarrow 0]{} -g'(x)$$

(sign is off!)

by continuity

So  $h$  is differentiable at  $x$  &  
 $h'(x) = -\frac{g'(x)}{(g(x))^2}$ .

Consequence: We get the full Power rule (with negative integer exponents!) [3]

Eg  $f(x) = x^{-3} = \frac{1}{x^3} \Rightarrow f'(x) = \frac{-g'(x)}{(g(x))^2} = \frac{-3x^2}{(x^3)^2} = -3x^{-4}$

In general:  $f(x) = x^{-n} = \frac{1}{x^n} \Rightarrow f'(x) = \frac{-nx^{n-1}}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-2}$   
So the same rule works for negative exponents!

Theorem 3: If  $h(x) = \frac{f(x)}{g(x)}$  &  $f, g$  are differentiable at  $x$  &  $g'(x) \neq 0$ ,

then  $h'(x) = f'(x) \frac{1}{g(x)} + f \cdot \left( -\frac{g'(x)}{(g(x))^2} \right) = \frac{f'_x g(x) - f_x g'(x)}{(g(x))^2}$ .

Example 1: Decide where  $f = \frac{x+1}{x-1}$  is defined,  
" continuous,  
is differentiable.

Soln: •  $f$  is defined when  $x-1 \neq 0$ , so for all  $x \neq 1$ .

•  $f$  is continuous for all  $x \neq 1$  (quotient of continuous functions  
& the denominator never vanishes in the domain of  $f$ )

Note:  $\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty$        $\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty$  so we cannot

extend  $f$  to  $x=1$  a continuous way.

• By the quotient rule,  $f$  is differentiable in its domain,

and  $f'(x) = \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} = \frac{1(x-1)+(x+1) \cdot 1}{(x-1)^2}$   
 $= \frac{2x}{(x-1)^2}$

In general: Quotient of polynomials  $\frac{P(x)}{Q(x)}$  is called a rational function.

• It is defined where  $Q(x) \neq 0$

• It is continuous in its domain b/c  $\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{\lim P(x)}{\lim Q(x)} = \frac{P(a)}{Q(a)}$  if  $Q(a) \neq 0$

Example 2: Find the equation of the tangent to  $y = \frac{x^3+x}{x-1}$  at  $(3, 10)$  & the normal.

Soh:  $f'(x) = \frac{(x^3+x)'(x-1) - (x^3+x)(x-1)'}{(x-1)^2} = \frac{(3x^2+1)(x-1) - (x^3+x)}{(x-1)^2}$

We want  $f'(2) = \frac{(3 \cdot 4 + 1) \cdot 1 - (8+2)}{1^2} = 13 - 10 = \boxed{3} = m_{\text{tan}}$ .

L<sub>tan</sub>:  $y = 3(x-2) + 10$

Recall: slope of normal =  $-\frac{1}{m_{\text{tan}}} = -\frac{1}{3}$  so

L<sub>norm</sub>:  $y = -\frac{1}{3}(x-2) + 10$ .