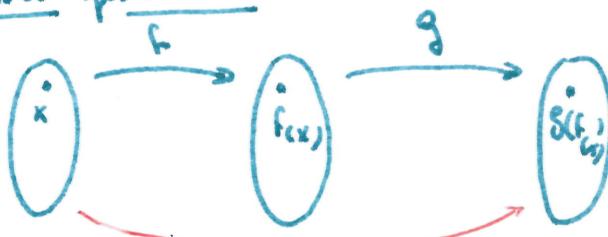


Lecture IX : § 3.3 Composite functions & the chain rule

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§1 Composite functions

Def



$$g \circ f(x) = g(f(x)) \quad \text{composite}$$

$$f: [a, b] \longrightarrow \mathbb{R}$$

$$g: \mathbb{R} \longrightarrow \mathbb{R}$$

$$\text{Composite } g \circ f: [a, b] \longrightarrow \mathbb{R}$$

is a new function on $[a, b]$ defined by $g \circ f(x) = g(f(x))$
 [first apply f & then apply g to $f(x)$]

Prop: If f is continuous at x_0 & g is continuous at $y = f(x_0)$, then $g \circ f$ is continuous at x_0 .

Proof: Enough to show: $\lim_{x \rightarrow x_0} g \circ f(x) = g \circ f(x_0)$. (see pictures on page 4)

Use ϵ/δ process: given $\epsilon > 0$, since g is continuous at $f(x_0)$, we know that we can find $\delta > 0$ so that if $0 < |y - y_0| < \delta$, then $|g(y) - g(y_0)| < \epsilon$. Now, take $\delta' = \delta > 0$ & consider only $y = f(x)$. Because f is cont at x_0 , we can find $\delta' > 0$ so that if $0 < |x - x_0| < \delta'$, then $|f(x) - f(x_0)| < \delta'$.

Now, take $y = f(x)$, so we know $0 < |f(x) - f(x_0)| < \delta' = \delta = \epsilon$ forces $|g(f(x)) - g(f(x_0))| < \epsilon$
 (if $f(x) = f(x_0)$, then $g(f(x)) = g(f(x_0))$ so it also holds that $|g(f(x)) - g(f(x_0))| < \epsilon$).

Answer: δ' works for ϵ . \blacksquare

Natural question: What can we say about differentiability?

Example: $f(x) = (x^3 + 4x)^{10}$ $g(y) = y^{10}$ $\Rightarrow g \circ f(x) = (x^3 + 4x)^{10}$

(clearly the 3 functions are differentiable. But computing $(g \circ f)'$ from the formula is hard: $\lim_{\Delta x \rightarrow 0} \frac{((x + \Delta x)^3 + 4(x + \Delta x))^{10} - (x^3 + 4x)^{10}}{\Delta x}$

(can use the Binomial Theorem to expand $()^{10}$ & cancel Δx as much as possible.)

Use the definition & assume f is differentiable at x & g is differentiable at $f(x)$. [2]

$$\begin{aligned}
 (g \circ f)'(x) &= \lim_{\Delta x \rightarrow 0} \frac{g(f(x + \Delta x)) - g(f(x))}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{g(\overset{\sim y+\Delta y}{f(x+\Delta x)}) - g(\overset{\sim y}{f(x)})}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{g(f(x + \Delta x)) - g(f(x))}{f(x + \Delta x) - f(x)} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x}
 \end{aligned}$$

If $\Delta x \rightarrow 0$ then $f(x + \Delta x) \rightarrow f(x)$, so $\Delta y = f(x + \Delta x) - f(x) \rightarrow 0$

Assume we have $f(x + \Delta x) \neq f(x)$ for Δx small enough; so $\Delta y \neq 0$.

$$\begin{aligned}
 &= \lim_{\substack{\Delta x \rightarrow 0 \\ (\Delta y \rightarrow 0)}} \frac{g(\cancel{f(x)} + \Delta y) - g(\cancel{f(x)})}{\Delta y} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 \text{Rewrite.} \quad &
 \end{aligned}$$

Warning = How do we know $\Delta y \neq 0$?

If in any neighborhood of x we have $f(x + \Delta x) = f(x)$, then $(R \circ f)'(x) = 0$ & $f'(x) = 0$.

Product of existing limits

$$\begin{aligned}
 &\uparrow \quad \downarrow \Delta y \rightarrow 0 \\
 &= \lim_{\Delta y \rightarrow 0} \frac{g(f(x) + \Delta y) - g(f(x))}{\Delta y} \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= g'(f(x)) \cdot f'(x).
 \end{aligned}$$

$g'(f(x))$
because g is diff'ble at $f(x)$

$f'(x)$
because f is diff'ble at x

Thm (Chain Rule) If f is differentiable at x & g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$

- Easy way to interpret this: "increment notation" (Leibniz Notation)

$$\frac{d(g \circ f)}{dx} = \frac{dg}{df} \frac{df}{dx}$$

$$\begin{aligned}
 \text{Example (above). } h(x) &= (x^3 + 4x)^{10} \Rightarrow h'(x) = 10 y^9 \Big|_{y=f(x)} \cdot (3x^2 + 4) \\
 g(y) &= y^{10}, \quad f(x) = x^3 + 4x \quad = 10 (x^3 + 4x)^9 \cdot (3x^2 + 4)
 \end{aligned}$$

In general $h(x) = f(x)^n$ for n integer : $h'(x) = n f(x)^{n-1} \cdot f'(x)$

Applications ① can deduce $\left(\frac{1}{f(x)}\right)'$. if f is differentiable & $f'(x) \neq 0$

$$\text{Write } h(x) = f(x)^{-1} \text{ & use Chain Rule } h'(x) = (-1)f(x)^{-2} f'(x)$$

$$= -\frac{f'(x)}{(f(x))^2} \text{ as we know!}$$

② can deduce the quotient rule:

$$\begin{aligned} \left(\frac{f(x)}{g(x)}\right)' &= \left(f(x) \cdot g(x)^{-1}\right)' \\ &\stackrel{\text{Prod Rule}}{=} f'(x) g(x)^{-1} + f(x) \left(g(x)^{-1}\right)' \\ &= f'(x) g(x)^{-1} + f(x) (-1) g(x)^{-2} g'(x) \\ &= \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2} \end{aligned}$$

So we don't need to memorize them!

Artin's Proof of Chain Rule = Avoids the issue that Δy could be 0.

Since $\frac{g(y+\Delta y) - g(y)}{\Delta y} \xrightarrow[\Delta y \rightarrow 0]{} g'(y)$, then the (RHS) is tiny near $\Delta y = 0$, call it ϵ .

$$\epsilon(\Delta y) := \frac{g(y+\Delta y) - g(y)}{\Delta y} - g'(y) \xrightarrow[\Delta y \rightarrow 0]{} 0$$

or equivalently: $g(y+\Delta y) - g(y) = \Delta y g'(y) + \Delta y \epsilon(\Delta y)$ with $\epsilon \xrightarrow[\Delta y \rightarrow 0]{} 0$

Now, $h = g \circ f$ satisfies:

$$\frac{h(x+\Delta x) - h(x)}{\Delta x} = \frac{g(y+\Delta y) - g(y)}{\Delta x} \quad (\text{write } \Delta y = f(x+\Delta x) - f(x))$$

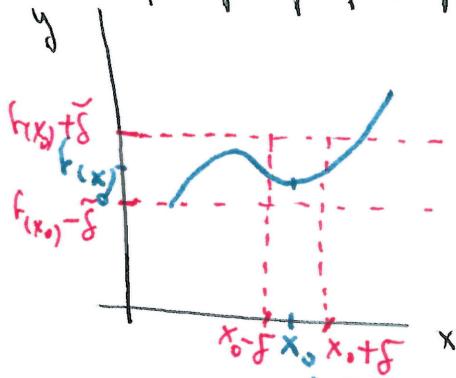
use (*) $\Rightarrow \frac{\Delta y}{\Delta x} g'(y) + \frac{\Delta y}{\Delta x} \epsilon(\Delta y)$ Note: we don't divide by Δy !

BUT: $\frac{\Delta y}{\Delta x} g'(y) \xrightarrow[\Delta x \rightarrow 0]{} f'(x) g'(f(x))$ & $\frac{\Delta y}{\Delta x} \epsilon(\Delta y) \xrightarrow[\Delta x \rightarrow 0]{} f'(x) \cdot 0$

because $\Delta y \xrightarrow[\Delta x \rightarrow 0]{} 0$ by the cont. of f . $\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = g'(f(x)) f'(x)$.

Pictures for proof of Prop in page 1.

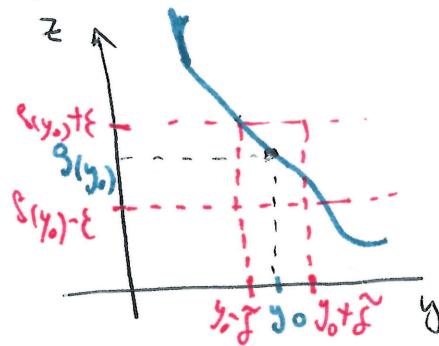
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graph of f

Use $\tilde{\delta}_>$ from g to
find $\delta > 0$

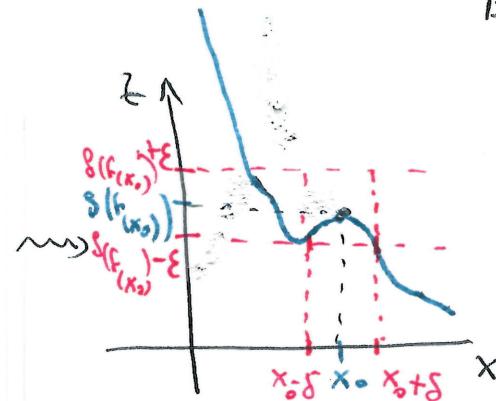
(We can do this because
 f is continuous in x_0)



graph of g

Find $\tilde{\delta}$ given $\varepsilon > 0$

(we can do this
because g is
continuous in $y = f(x_0)$)



graph of $g \circ f$

want to find $\delta = \delta(\varepsilon)$
given $\varepsilon > 0$

We find δ in 2 steps!

Exercise: Find the explicit $\tilde{\delta} & \delta$ for the functions:

$$\begin{cases} g(x) = x^2 + 5 \\ f(x) = 2x + 1 \end{cases} \quad \text{(Lecture III)}$$

Soln: $\tilde{\delta} = \tilde{\delta}(\varepsilon) = \sqrt{\varepsilon}$ $\left\{ \begin{array}{l} \tilde{\delta} = \frac{\varepsilon}{2} \\ \delta = \frac{\sqrt{\varepsilon}}{2} \end{array} \right.$