

Lecture X: § 3.4 Some trigonometric derivatives

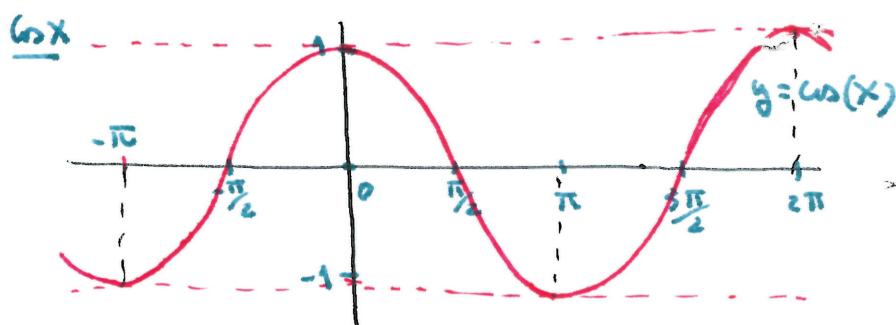
So far: derivation rules for addition, scalar mult., multiplication, quotient
 $f(x) + g(x)$ $c \cdot f(x)$ $f(x) \cdot g(x)$ $\frac{f(x)}{g(x)}$

$$\text{composition (chain rule): } (g \circ f)'_{(x)} = g'(f(x)) \cdot f'(x).$$

→ Derivatives of polynomials, powers of polynomials

Q: What about other functions? Can we have more building blocks than x^n ?
§ 1 Trigonometric functions:

2 Basic functions: $\sin(x)$ & $\cos(x)$ $[\tan(x) = \frac{\sin x}{\cos x}]$

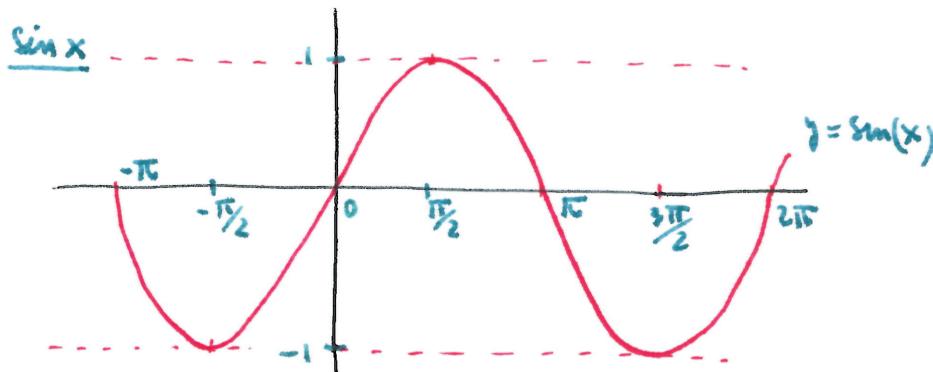


Useful Properties

- $\cos x$ is periodic: $(\cos(x+2\pi)) = \cos x$
- $\cos(-x) = \cos x$ (Even)
- $-1 \leq \cos x \leq 1$ for all x

Useful Values:

$$\begin{aligned} \cos(0) &= 1 & \cos(\frac{\pi}{3}) &= \frac{1}{2} \\ \cos(\pi) &= -1 & \cos(\frac{\pi}{6}) &= \frac{\sqrt{3}}{2} \\ \cos(\frac{\pi}{2}) &= 0 & \cos(\frac{3\pi}{2}) &= 0 \\ \cos(\frac{\pi}{4}) &= \frac{\sqrt{2}}{2} & & \end{aligned}$$



Properties

- $\sin(x)$ is periodic w/ period 2π
- $\sin(x)$ is odd: $\sin(-x) = -\sin x$
- $-1 \leq \sin x \leq 1$ for all x

Useful Values:

$$\begin{aligned} \sin(0) &= \sin(\pi) = 0 & \sin(\frac{\pi}{3}) &= \frac{\sqrt{3}}{2} \\ \sin(\frac{\pi}{2}) &= 1 & \sin(\frac{\pi}{6}) &= \frac{1}{2} \\ \sin(\frac{3\pi}{2}) &= -1 & & \\ \sin(\frac{\pi}{4}) &= \frac{\sqrt{2}}{2} & & \end{aligned}$$

graph is a shift of $\cos x$: $\sin(x) = \cos(x - \frac{\pi}{2})$

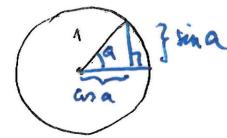
These 2 functions are nice and smooth, so they should be differentiable.
 We'll find the formulas by computing the increments $\sin(x + \Delta x) - \sin x$
 $\cos(x + \Delta x) - \cos x$, so we'll need formulas for $\sin x$ & $\cos x$:
 addition

$$(1) \quad \sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$(2) \quad \cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

Also: Basic equation
 (from Pythagorean Thm)

$$\sin^2 x + \cos^2 x = 1$$



Theorem: (i) $\frac{d}{dx} \sin x = \cos x$ (ii) $\frac{d}{dx} \cos x = -\sin x$.

Proof: Use the definition of derivatives.

$$\begin{aligned}
 (i) \frac{d}{dx} \sin x &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(\sin x \cos \Delta x + \sin \Delta x \cos x) - \sin x}{\Delta x} \\
 &\stackrel{\text{Addition formula (1)}}{=} \lim_{\Delta x \rightarrow 0} \left(\sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \frac{\sin \Delta x}{\Delta x} \cos x \right) \\
 &\text{rearrange} \\
 &\stackrel{\omega x = 0 + \Delta x}{=} \lim_{\Delta x \rightarrow 0} \frac{\sin x \left(\frac{\cos(\omega x + \Delta x) - \cos \omega x}{\Delta x} \right)}{\sin x} - \frac{\frac{\sin \Delta x}{\Delta x}}{\frac{\cos x}{\cos x}} \\
 &\quad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \\
 &\quad ? = \frac{d \cos x}{dx}(0) \text{ What is this?}
 \end{aligned}$$

$$\begin{aligned}
 (\text{Claim}) : \lim_{\Delta x \rightarrow 0} \frac{\cos \Delta x - 1}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\cos \Delta x - 1}{\Delta x} \cdot \frac{\cos \Delta x + 1}{\cos \Delta x + 1} = \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\cos^2 \Delta x - 1}{\Delta x (\cos \Delta x + 1)} = \lim_{\Delta x \rightarrow 0} \frac{\sin^2 \Delta x}{\Delta x (\cos \Delta x + 1)}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{\sin \Delta x}{\Delta x}}{\frac{\cos \Delta x + 1}{\cos \Delta x + 1}} \stackrel{\Delta x \rightarrow 0}{\rightarrow} 0 \quad (\text{since } \cos x \text{ is cont}) \\
 &\quad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \\
 &= 1 \cdot \frac{0}{2} = 0
 \end{aligned}$$

$$\text{Conclusion : } \frac{d}{dx} \sin x = \sin(x) \cdot 0 + 1 \cos x = \boxed{+\cos x} \quad \checkmark$$

$$(ii) \frac{d}{dx} \cos x = \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$\begin{aligned}
 &\stackrel{\text{rearrange}}{=} \lim_{\Delta x \rightarrow 0} \left(\cos x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) - \frac{\sin \Delta x}{\Delta x} \sin x \right) \\
 &\quad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \qquad \downarrow \Delta x \rightarrow 0 \\
 &\quad \cos x \cdot 0 - 1 \cdot \sin x \\
 &\quad \text{Product rule for limits} = \boxed{-\sin x} \quad \checkmark
 \end{aligned}$$

Q: What about the other trig functions?

$$\textcircled{1} \quad \tan(x) = \frac{\sin(x)}{\cos(x)} \quad \rightsquigarrow \text{defined for all } x \neq \frac{(2k+1)\pi}{2} \text{ with } k \text{ integer}$$

(because we
need $\cos x \neq 0$)

Use the Quotient Rule!

$$\begin{aligned} \frac{d}{dx} \tan x &= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} \\ &= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

//
 $\cos^2 x + \sin^2 x = 1$

So

$$\boxed{\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}}$$

$$= \sec^2(x)$$

$$\textcircled{2} \quad \cot(x) = \frac{1}{\tan(x)} = \frac{\sin(x)}{\cos(x)} \quad \rightsquigarrow \text{use Quot Rule!} \quad \text{Defined to all } x \neq k\pi \quad (\text{since } \sin x \neq 0)$$

$$\frac{d}{dx} \cot(x) = \frac{-(\tan x)'}{\tan^2(x)} = -\frac{1}{\cos^2(x)} \cdot \frac{\cos^2(x)}{\sin^2(x)} = \boxed{-\frac{1}{\sin^2(x)}} = -\csc^2(x)$$

for k integer

$$\textcircled{3} \quad \sec(x) = \frac{1}{\cos(x)} \quad \rightsquigarrow \quad \frac{d}{dx} \sec x = \frac{-(-\sin(x))}{\cos^2(x)} = \boxed{\frac{\sin(x)}{\cos^2(x)}} = \tan^2 x \sec x$$

defined for $x \neq (2k+1)\frac{\pi}{2}$ for k integer

$$\textcircled{4} \quad \csc(x) = \frac{1}{\sin(x)} \quad \rightsquigarrow \quad \frac{d}{dx} \csc x = \boxed{\frac{-(\cos x)}{\sin^2(x)}} = -\cot^2 x \cdot \csc(x)$$

defined for $x \neq k\pi$ for k integer.

\triangle We don't need to memorize these formulas, we just use Quotient Rule to get them from the Thm!

Examples:

$$(1) f(x) = \sin^3(x) \quad \rightsquigarrow \quad f'(x) \stackrel{\text{Chain Rule}}{=} 3 \sin^2(x) \cdot (\sin(x))' = 3 \sin^2 x \cos x$$

$$(2) f(x) = \sin(5x) \quad \rightsquigarrow \quad f'(x) \stackrel{\text{Chain Rule}}{=} \cos(5x) \cdot 5$$

$$(3) f(x) = \cos((4x+1)^2) \stackrel{\text{Chain Rule}}{=} -\sin((4x+1)^2) \cdot \frac{d}{dx}((4x+1)^2)$$

$$= -\sin((4x+1)^2) \cdot 2(4x+1) \cdot 4 = -8(4x+1) \sin((4x+1)^2)$$

$$(4) f(x) = \sin^2 x \cos^3 x \stackrel{\text{Prod Rule}}{=} (\sin^2 x)' \cos^3(x) + \sin^2 x (\cos^3 x)'$$

$$\text{Chain Rule} \quad \stackrel{\text{Chain Rule}}{=} 2 \sin(x) \cos(x) \cos^3(x) \stackrel{\text{Chain Rule}}{=} 2 \sin^2(x) \cdot 3 \cos^2(x) (-\sin x) = \boxed{\sin^2 x \sin x (-3 + 5 \cos^2 x)}$$