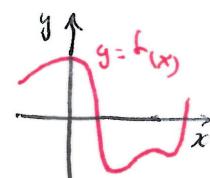


Lecture XI: 5.3.5 Implicit functions & fractional exponents

So far we've treated functions $f: D \rightarrow \mathbb{R}$ given by a formula

$$\text{E.g. } y = f(x) = (x^3 + 4x)^{10} \quad \text{or} \quad y = \sin(x)$$

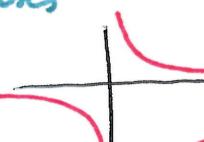


This gives a curve in the plane = the graph of the function f

Often times, we deal with curves given by a relation between the independent variable x & the dependent variable y , and we cannot solve for $y = f(x)$.

Examples: Classical plane curves

$$1. \text{ Hyperbola: } xy = 1$$



graph of $y = \frac{1}{x}$ in $\mathbb{R} - \{0\}$

$$2. \text{ Circle centered at } (0,0): x^2 + y^2 = R^2$$



NOT the graph of a function because vertical test fails!

$$3. \text{ Ellipse: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a, b, c > 0.$$

$$4. \quad x^2 - y^2 = R \quad \text{again a Hyperbola.}$$

NOTE: From $x^2 + y^2 = R^2$ we have 2 possible solutions $y = \pm \sqrt{R^2 - x^2}$.
The curve is the union of two half-circles (one per sign!) $\Rightarrow -R \leq x \leq R$.

$$5. \text{ Equation: } 2y^2 - 2xy = 10 - x^2 \Rightarrow 2y^2 - 2xy + x^2 = 10$$

now solve for y with quadratic formula

$$y = \frac{2x \pm \sqrt{4x^2 - 4 \cdot 2 \cdot (x^2 - 10)}}{2 \cdot 2} = \frac{2x \pm \sqrt{80 - 4x^2}}{4}$$

$$y = \frac{x \pm \sqrt{20 - x^2}}{2}. \quad \text{again: the curve is the union of two}$$

NOTE: Can "solve by radicals" only if the degree in y is ≤ 2 . graphs $f_+, f_- : [-\sqrt{20}, +\sqrt{20}] \rightarrow \mathbb{R}$.

Q: Can we compute $\frac{dy}{dx}$ without the explicit formula, but just using the implicit equation relating y & x ?

A: Yes! \Rightarrow implicit differentiation.

Guiding principles: "If 2 functions are equal, so are their derivatives"
Chain rule & various techniques for computing derivatives

$$\text{Eg1 } 2y^2 - 2xy = 10 - x^2 \quad (\star) \text{ Think } y = y(x)$$

Take $\frac{d}{dx}$ on both sides & use chain rule & product rule

$$2 \cdot 2y \cdot \frac{dy}{dx} - 2(y + x \frac{dy}{dx}) = -2x$$

$$4y \boxed{\frac{dy}{dx}} - 2y - 2x \boxed{\frac{dy}{dx}} = -2x \quad \Rightarrow \text{ Try to solve for } \frac{dy}{dx}.$$

Common factor $\frac{dy}{dx}$ $(4y - 2x) \frac{dy}{dx} = -2x + 2y$

$$(2y - x) \frac{dy}{dx} = y - x \quad \Rightarrow \boxed{\frac{dy}{dx} = \frac{y-x}{2y-x}} \quad (1)$$

This works assuming $2y \neq x$. We need to check this with the original equation (\star) , by replacing $y = \frac{x}{2}$ \Rightarrow This gives the values of x we should avoid!

$$2y^2 - 2xy = 2 \cdot \frac{x^2}{4} - 2x \cdot \frac{x}{2} = \frac{x^2}{2} - x^2 = \frac{-x^2}{2} \stackrel{?}{=} 10 - x^2$$

$$x^2 = 20 \Leftrightarrow \frac{1}{2}x^2 = 10$$

These are the only points that we need to avoid for the formula (1) for $\frac{dy}{dx}$ to hold.

Check that (1) works for f_+ & f_- .

$$\text{For } f_+ \frac{dy}{dx} = \frac{1}{2} + \frac{1}{2} \frac{d}{dx} \sqrt{20-x^2} \stackrel{\text{later}}{=} \frac{1}{2} + \frac{1}{4} \frac{-2x}{\sqrt{20-x^2}}$$

$$\text{Use } \sqrt{20-x^2} = 2y-x \Rightarrow \frac{dy}{dx} = \frac{1}{2} + \frac{x}{2(2y-x)} = \frac{2y-x-x}{2(2y-x)} = \frac{y-x}{2y-x} \checkmark$$

Similar method verifies the formula for f_- .

$$\text{Eg2: } x^2 + y^2 = R^2 \Rightarrow \text{Implicit Diff. } 2x + 2y \frac{dy}{dx} = 0$$

Again, this is valid if $y \neq 0$, that is $x^2 \neq R^2$ so $x \neq \pm \sqrt{R}$

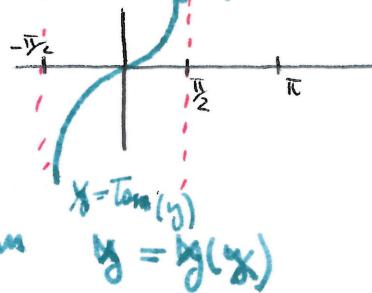
Again, we can double-check this if $y = \pm \sqrt{R^2 - x^2}$ (using $y=0$ in the original equation)

$$y' = \pm \frac{1}{2} \frac{(-2x)}{\sqrt{R^2 - x^2}} = -\frac{x}{\pm \sqrt{R^2 - x^2}} = -\frac{x}{y} .$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Application 1 Derivative of inverse trig functions :

Eg : $\tan(y) = x$



has an inverse (it's bijection!)

Use implicit differentiation

$$\frac{d}{dy} \tan(y) \cdot \frac{dy}{dx} = 1$$

$$\sec^2 y \cdot \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cot^2 y \rightsquigarrow \text{still not explicit enough!}$$

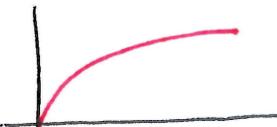
$$\text{We can go even further: } x^2 = \tan^2 y = \frac{\sec^2 y}{\cot^2 y} = \frac{1 - \cot^2 y}{\cot^2 y} = \frac{1}{\cot^2 y} - 1$$

$$\text{so } \frac{1}{\cot^2 y} = x^2 + 1, \text{ ie } \cot^2 y = \frac{1}{x^2 + 1} \quad \text{We get}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x^2 + 1}}$$

Application 2 : Fractional Exponents

$$y = x^{\frac{p}{q}} \quad \text{for } p, q \text{ integers } q \neq 0 \quad \text{means} \quad y^q = x^p$$

Eg  , is $y = x^{\frac{1}{2}}$ $\rightsquigarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ because $y^2 = x$ gives $2yy' = 1 \rightsquigarrow y' = \frac{1}{2y}$

Use implicit differentiation : $\frac{q}{q-1} y^{\frac{q-1}{q}} \frac{dy}{dx} = p x^{p-1}$ or $\frac{dy}{dx} = \frac{p}{q} x^{\frac{p-1}{q-1}}$ (By power rule)

$$\text{But } y^q = x^p \text{ so } y^{q-1} = x^{p-1} y^{-1} = \frac{x^p}{y} = \frac{x^p}{x^{\frac{p}{q}}} = x^{p - \frac{p}{q}}$$

$$\text{Then } \frac{dy}{dx} = \frac{p}{q} x^{p-1 - \frac{p}{q}} = \boxed{\frac{p}{q} x^{\frac{p}{q}-1}}$$

\rightsquigarrow same Power Rule works with fractional exponents!

Eg : $y = \sqrt{w\sin x} \rightsquigarrow y' = \frac{1}{2} \frac{1}{\sqrt{w\sin x}} (w\sin x)' = \frac{-w\cos x}{2\sqrt{w\sin x}}$