

§1 Derivatives of Higher Order:

Simple idea: If $y = f(x)$ is differentiable, then $y' = f'(x)$ is a function & it might be differentiable. If so, we can differentiate again and write $(y')' = (f')'(x) = f''(x)$, and so on...

Notation: y'' , $f''(x)$, $f^{(2)}(x)$, $\frac{d}{dx} \left(\frac{d}{dx} f \right) = \frac{d^2 f}{dx^2}$, $D^2 f$
 y''' , $f'''(x)$, $f^{(3)}(x)$, $\frac{d^3 f}{dx^3}$, $D^3 f$

In general: $y^{(n)}$, $f^{(n)}(x)$, $\frac{d^n f}{dx^n}$, $D^n f$ for $n \geq 0$ integer

Convention: $f^{(0)}(x) = f$.

Example 1: Polynomials \Rightarrow enough to work out monomials

$y = x^n$ for $n > 0$ integer, c constant $\Rightarrow c' = 0$, $c'' = 0, \dots$
 $c^{(n)} = 0$ for $n > 0$

$y' = n x^{n-1}$, $y'' = n(n-1)x^{n-2}$, $y^{(3)} = n(n-1)(n-2)x^{n-3}$,

When does this stop? $y^{(k)} = n(n-1)\dots(n-k+1)x^{n-k}$ for $k \leq n$

$y^{(n+1)} = 0$, so $y^{(k)} = 0$ for $k > n$.

Note: Can express these with $p! = p(p-1)\dots 2 \cdot 1$ ("p factorial") for $p \geq 1$ integer: $y^{(k)} = \frac{n!}{(n-k)!} x^{n-k}$ if $k \leq n$

Example 2: $y = x^{-n} = \frac{1}{x^n}$ for $n > 0$ integer

$y' = -\frac{n}{x^{n+1}}$, $y'' = \frac{n(n+1)}{x^{n+2}}$, $y''' = -\frac{n(n+1)(n+2)}{x^{n+3}}$, so never ends!

In general: $y^{(k)} = (-1)^k \frac{n(n+1)\dots(n+k-1)}{x^{n+k}}$ for all $k \geq 0$
 $= \frac{(n+k-1)!}{(n-1)!} \frac{(-1)^k}{x^{n+k}}$ integer.

Example 3: Trig functions

$y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$, $y^{(4)} = \sin x$

& it repeats from here. Similar behavior for cosine.

Remark: $\sin(x)$ & $\cos(x)$ will both satisfy $y'' = -y$
 Only solutions to the ODE $y'' = -y$ are $y = a\sin(x) + b\cos(x)$ for
 ("Linear combinations" of $\sin(x)$ & $\cos(x)$) of the form $a, b \in \mathbb{R}$

Remark: We can combine higher derivatives with implicit differentiation

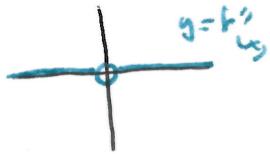
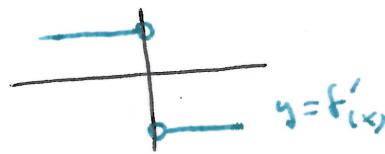
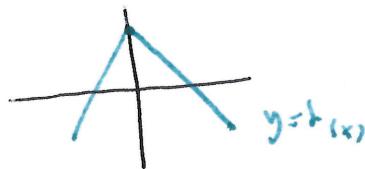
$$\text{Eg: } x^2 + y^2 = R^2 \implies 2x + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y} \text{ diff'ble}$$

$$\text{Try again: } 2 + 2\left(\frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}\right) = 0$$

$$\text{use } \frac{dy}{dx} = -\frac{x}{y} \implies 1 + \left(\left(-\frac{x}{y}\right)^2 + y \frac{d^2y}{dx^2}\right) = 0, \text{ so } \frac{d^2y}{dx^2} = \frac{-1 - \frac{x^2}{y^2}}{y}$$

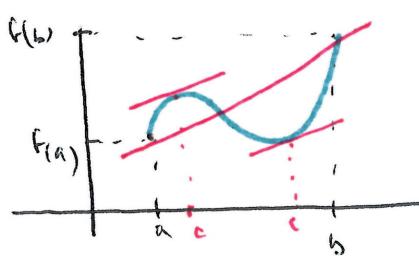
$$\text{get, } \frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3} = \boxed{\frac{-R^2}{y^3}}$$

$$\text{Eg: } f(x) = 1 - |x| = \begin{cases} 1-x & x \geq 0 \\ 1+x & x < 0 \end{cases} \rightarrow f'(x) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases} \implies f''(x) = 0 \quad x \neq 0 \\ \text{not diff'ble at } x=0 \quad \text{diff'ble away from } x=0 \quad \text{undefined at } x=0$$



3.2 Appendix A4: Proof of Mean Value Theorem.

Thm (MVT): If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ & differentiable on (a, b) , there exists c in (a, b) with $f'(c) = \frac{f(b) - f(a)}{b - a}$



we show this result in a special case: $f(a) = f(b)$

Rolle's Thm: If $f: (a, b) \rightarrow \mathbb{R}$ is cont on $[a, b]$, differentiable on (a, b) & $f(b) = f(a)$, then we can find c in (a, b) with $f'(c) = 0$

Proof: By the Extreme Values Thm we know f achieves its max & min values on $[a, b]$. Unless f is constant (and so $f' \equiv 0$ on (a, b)), we can assume f is not constantly $= f(a)$, so we can find x with $f(x) > f(a)$

(so the max is $> f(a)$ & it's achieved at some c in (a, b)) or
 $f(x) < f(a)$ (so the min is $< f(a)$ & it's achieved at some c in (a, b)).
 Then, there is an extreme value achieved for some $a < c < b$ &
 by Lecture VII, we know $f'(c) = 0$. \blacksquare

We now use Rolle's Thm to prove MVT.

Proof of MVT: Need to write an auxiliary function that satisfies the hypotheses of Rolle's Thm.

The eqn of the secant line through $(a, f(a))$ & $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a} (x - a)$$

We take $g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right)$ for $a \leq x \leq b$

By construction: • g is cont in $[a, b]$
 • $g(x)$ is differentiable in (a, b)
 • $g(a) = 0$, $g(b) = 0$ } \Rightarrow By Rolle's Thm, there is $a < c < b$ with $g'(c) = 0$

But $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$, so $g'(c) = 0$ gives $f'(c) = \frac{f(b) - f(a)}{b - a}$ \blacksquare

We end with a generalization of MVT:

Thm (Generalized MVT) Pick $f, g : [a, b] \rightarrow \mathbb{R}$ continuous functions in $[a, b]$ & diff'ble in (a, b) . Assume $g'(x) \neq 0$ for all x in (a, b) . Then there exists at least one c in (a, b) with $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

Note: $g(x) = x$ gives back MVT.

Proof: We claim $g(a) \neq g(b)$. Otherwise, by Rolle's Thm $g'(c) = 0$ for some $a < c < b$, contradicting w/ the assumptions of the Thm. We consider the auxiliary function:

$$h(x) = (f(b) - f(a)) (\underline{g(x)} - g(a)) - (f(x) - f(a)) (g(b) - g(a))$$

By constr. • h is cont in $[a, b]$ & $h'(x) = (f(b) - f(a)) g'(x) - (g(b) - g(a)) f'(x)$
 • $h(a) = 0 = h(b)$ | Thus, by Rolle's Thm $h'(c) = 0$ for some $a < c < b$. This proves the Thm \blacksquare