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Lecture XIII: Appendix A3: The Extreme Value Thm

. § 9.1: Increasing & Decreasing Functions: Max & Min

Last Time: We used the Extreme Value Thm to prove the Mean Value Thm (a Rolle's Thm).
But we still don't know why GVT holds!

§ 1 The Extreme Value Thm:

To prove the result, we need an auxiliary statement:

Boundedness Thm: If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, then it is bounded, meaning there exists a number C satisfying $|f(x)| \leq C$ for all $a \leq x \leq b$.

Note: $f(x) = \frac{1}{x}$ is not bounded on $[0, 1]$, & we cannot extend f to $[0, 1]$ in a continuous way because $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$. ($f(\frac{1}{n}) = n \xrightarrow{n \rightarrow \infty} +\infty$)

Proof: We need a property of real numbers (one of the axioms defining \mathbb{R})

Prop: "Any nonempty set of real numbers with an upper bound has a least upper bound"

($S \neq \emptyset$ with upper bound M , means $s \leq M$ for all $s \in S$.)

\tilde{M} is a least upper bound if \tilde{M} is an upper bound & for any $M < \tilde{M}$ we can find $s \in S$ with $M < s \leq \tilde{M}$.)

Eg: $S = \left\{ 1 - \frac{1}{n} : n \geq 0 \text{ integer} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

1 is an upper bound. $1 - \frac{1}{n} \leq 1$ for all $n \geq 1$.

1 is the least upper bound $\xrightarrow{\text{NOTE: } 1 \notin S}$ (least upper bound does not belong to S necessarily)

Define $S = \{ c \text{ in } [a, b] : f \text{ is bounded on } [a, c] \}$

S is non-empty because $a \in S$ (f is bounded in $\{a\} = [a, a]$)

b is an upper bound for S

By the defining property of \mathbb{R} , S has a least upper bound. Call it c_0 .

Claim: $c_0 = b$.

Assume this is not true, so $c_0 < b$. Since f is cont at $x = c_0$ we can

find $\delta > 0$ where if $0 < |x - c_0| < \delta$ then $|f(x) - f(c_0)| < 1$.

In particular $-1 + f(c_0) < f(x) < 1 + f(c_0)$ so $f(x)$ is bounded in $(c_0 - \delta, c_0 + \delta)$

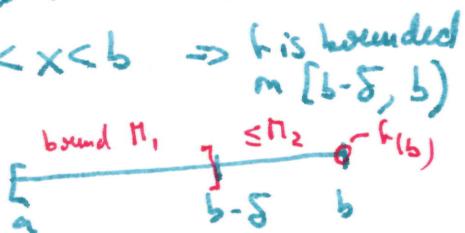
But then f is bounded on $[a, c_0]$
 $f \dots$ on $[c_0, c_0 + \frac{\delta}{2}]$ } so f is bounded on $[a, c_0 + \frac{\delta}{2}]$
 $\& c_0 + \frac{\delta}{2} < b$.

But then $c_0 + \frac{\delta}{2}$ lies in S , contradicting the fact that c_0 was the least upper bound for S .

This forces $c_0 = b$ and we are left with f is bounded on every interval $[a, c]$ for $c < b$.

Claim: $b \in S$ Again, by the cut of f at $x=b$, we can find $\delta > 0$ with $|f(x) - f(b)| < 1$ whenever $b-\delta < x < b$. Then f is bounded

Then . $f(b)-1 < f(c) < f(b)+1$ if $b-\delta < c < b \Rightarrow f$ is bounded
 . f is bounded on $[a, b-\delta]$
 . f is bounded on $\{b\}$



So f is bounded on $[a, b]$. \square

Q: Then we make sure the bound is attained?

Extreme Value Thm: $f: [a, b] \rightarrow \mathbb{R}$ continuous, then f achieves a max & a minimal value on $[a, b]$, that is, there exist c, d in $[a, b]$ with $f(c) \leq f(x) \leq f(d)$ for all x in $[a, b]$

Proof: We argue by contradiction to show the number d exists.

By the Boundedness Thm we know there exists M with $f(x) \leq M$ for all x

We can take M to be the least upper bound of the image of f .

If d does not exist, this means $f(x) < M$ for all x in $[a, b]$. Consider

the new function $g(x) = \frac{1}{M-f(x)}$.

g is cont in $[a, b]$ & $g(x) > 0$ on $[a, b]$

By the Boundedness Thm, we can find $C > 0$ with $0 < g(x) = \frac{1}{M-f(x)} < C$
 for all x in $[a, b]$.

But then: $1 < C(M-f(x))$, or $f(x) < \frac{CM-1}{C} = M - \frac{1}{C}$

But M was the least upper bound & $M - \frac{1}{C} < M$. This is a contradiction!

We conclude the the maximum of f is achieved

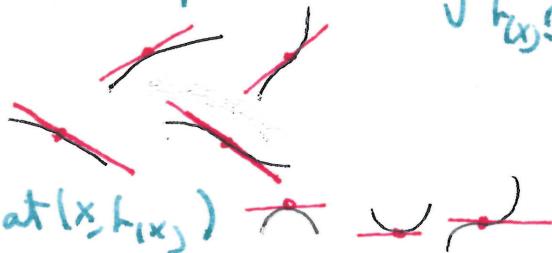
Similar proof works for showing that the minimum value is achieved \square

§2 Increasing & Decreasing functions: Maxima & Minima

GOAL: Use f' (or higher order derivatives) to sketch the graph of $f(x)$.

Q: Assume f is differentiable. What does f' tell us quantitatively about $f(x)$?

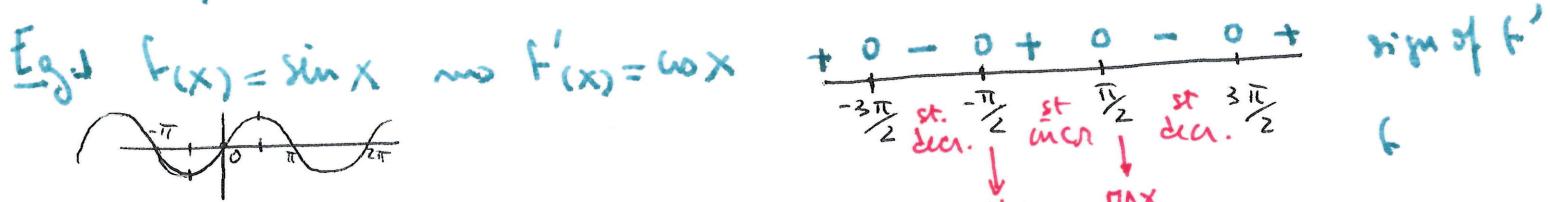
Eg: $f'(x) > 0 \Rightarrow f$ strictly increasing



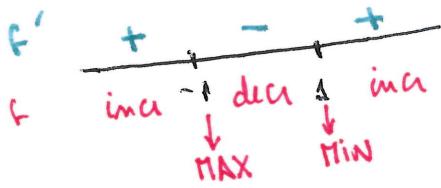
$f'(x) < 0 \Rightarrow f$ strictly decreasing

$f'(x) = 0 \Rightarrow f$ has a horiz. tangent at $(x, f(x))$

So if we can find regions where we know the sign of $f'(x)$, we can make qualitative statements about the growth of $f(x)$.



Eg 2: $f(t) = t^5 - 5t + 1 \Rightarrow f'(t) = 5t^4 - 5 = 5(t^4 - 1) = 5(t^2 + 1)(t^2 - 1) > 0$



Also $\lim_{t \rightarrow +\infty} f(t) = +\infty$
 $\lim_{t \rightarrow -\infty} f(t) = -\infty$

$\Rightarrow f$ has no extreme values in \mathbb{R}

Useful things to determine:

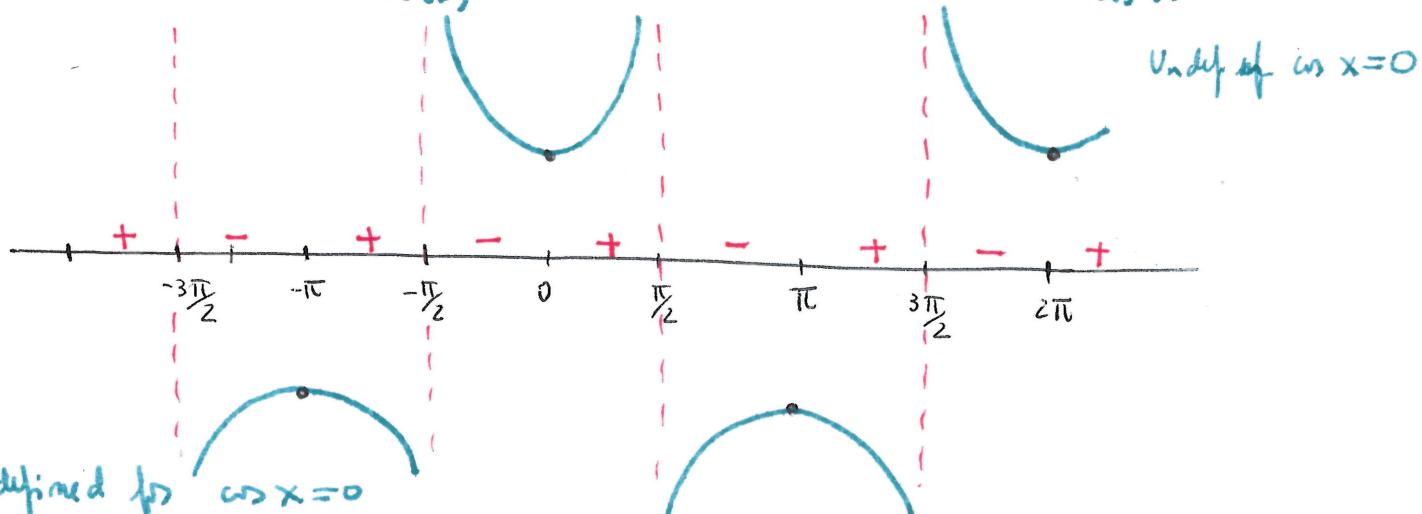
1. Critical Points = either $f'(x) = 0$ or $f'(x)$ is undefined (eg $f(x) = |x|$
 $x=0$ is crit pt)
2. Critical Values = $f(x)$ for x a critical pt.
3. Sign of $f'(x)$ between critical pts & between undefined pts
 $(\text{eg } f(x) = \frac{1}{x(x+1)} \text{ for } x \neq 0, -1)$
4. Intercepts: $f(0)$ & $f(x) = 0$
 $(y\text{-intercept}) \quad (x\text{-intercepts})$
5. Behavior of f near $\pm\infty$ ($\lim_{x \rightarrow +\infty} f(x)$ & $\lim_{x \rightarrow -\infty} f(x)$)
 \Rightarrow Potential asymptotes.
6. Behavior of f near the pts where f is not defined (eg $f(x) = \frac{1}{x(x+1)}$ for $x \neq 0, -1$)

Note: Finding x -intercepts in Ex 2 is hard! (we can only guess: $f(0) = 1$ by IVT
 $f(-1) = -3$ by IVT & number of zeros: $f(1) = -3$)

We have a zero in $(0, 1)$. $f(-1) = 5 > 0$
Last zero in $(1, 2)$. $f(-2) = -21 < 0$ we have a zero in $(-2, -1)$. Only 3 zeros!

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Ex: $f(x) = \sec(x) = \frac{1}{\cos(x)}$ $\Rightarrow f'(x) = \sec(x) \tan(x) = \frac{\sin x}{\cos^2 x}$



0. Undefined for $\cos x = 0$
1. Crit pts: $\cos x = 0$ (but f is also undef) or $\sin x = 0 \rightarrow x = k\pi$ for k integer
2. Crit Values $f(k\pi) = \frac{1}{\cos(k\pi)} = \pm 1$ ($+1$ for k even, -1 " k odd)
3. Sign between consecutive multiples of $\frac{\pi}{2}$ \leftrightarrow sign of $\sin(x)$ in those intervals
4. Zeros of $f'(x)$: none!
5. Behavior at $\pm\infty$: no limit exists because f is periodic!
6. Behavior near $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ } Side limits are $\pm\infty$.
 $-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \dots$ } [Vertical Asymptotes]