

Lecture XIV: § 4.2 Concavity & points of inflection

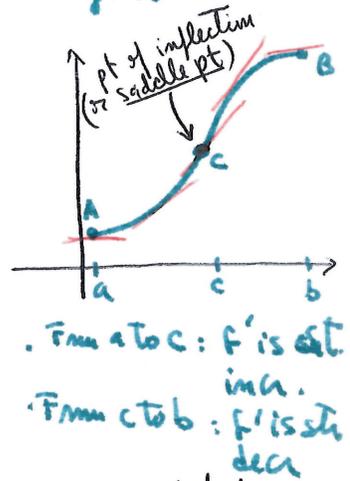
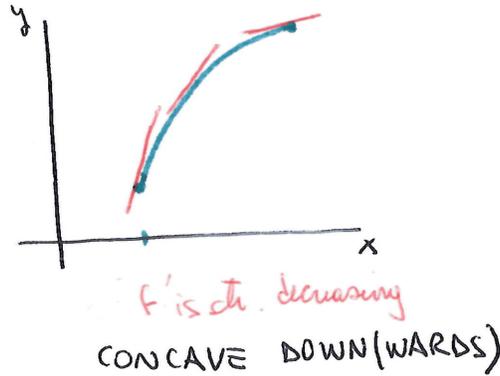
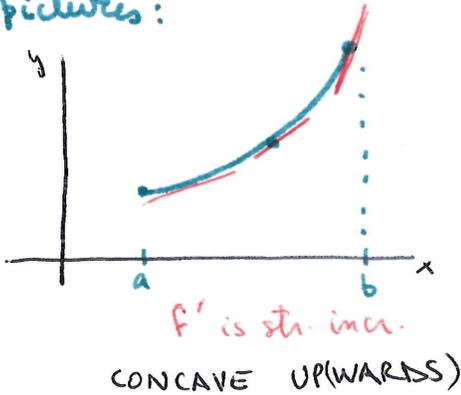
Last time: Used f' to study growth of f , local extrema & extreme values

TODAY: Use higher order derivatives to study concavity = bending of the graph of f

Key $f'' = (f')'$ gives us qualitative information about f' .

Guiding principle: f'' will give info on the growth of f' = slope of tangent lines!

• 3 pictures:



Formally:

Def.: If the graph of f lies above all of its tangent lines on the interval $[a, b]$, we say f is concave up(wards) on $[a, b]$

• If the graph of f lies below all of its tangent lines on the interval $[a, b]$ we say f is concave down(wards) on $[a, b]$

Q: How to test this without drawing?

Concavity Test: Assume f' is differentiable on (a, b)

(1) If $f'' > 0$ on (a, b) then f is concave up on $[a, b]$ (write C.U.)

(2) If $f'' < 0$ _____ down on $[a, b]$ (write C.D.)

Why? On (1) f' is increasing & on (2) f' is decreasing (see page 4)

Example $f(x) = x^3$. Find intervals where f is CU / CD.

$$f'(x) = 3x^2, \quad f''(x) = 6x \quad \begin{array}{c} f'' \\ f \end{array} \begin{array}{c} - \\ x=0 \\ + \end{array} \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \\ \\ \end{array}$$

At $x=0$, the tangent line is horizontal & graph lies on both sides of the line \Rightarrow change in concavity at $x=0$!

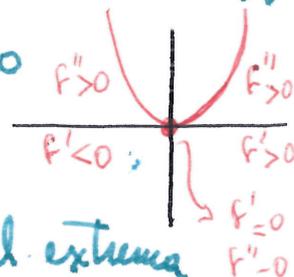
Def: A point c in the domain of f is an inflection point if f is continuous at c & the function changes from CU to CD , or from CD to CU at c . (At these points, the direction of concavity changes)

Remark: Inflection points satisfy: $f''(x)=0$ or f'' is not defined at x [like it happened with crit. points]. But local max/min also can have $f''(x)=0$

In short: $f''(x)=0$ is not a sufficient condition to be an inf. pt.

Example: $f(x) = x^4$ has a (local) minimum at $x=0$

& $f'(x) = 4x^3$, $f''(x) = 12x^2$ so $f''(0) = 0$



. This example leads to the following criterion to find local extrema

The Second Derivative Test Suppose $f''(x)$ is continuous near $x=c$

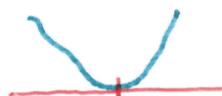
(1) If $f'(c)=0$ and $f''(c) > 0$, then f has a local min at c

(2) If $f'(c)=0$ and $f''(c) < 0$ _____ max at c

Why? $f'(c)=0$ says that tangent is horizontal.

. If $f''(c) > 0$, since f'' is cont at c , then around c , $f''(x) > 0$ for all x in a small interval

So f'' is CU in this interval which means the graph is above the tangent line at $(c, f(c))$



So c is a local min for f .

. The same arguments hold for (2). ■

Warning The test CANNOT be used when $f''(c) = 0$. The test fails when $f''(c)$ does not exist.

Example: Find the local max/min & inflection points of

$f(x) = 1 + 3x^2 - 2x^3$ \rightarrow diff'ble everywhere, so local max/min are crit pts with $f'(x)=0$

Soln: $f'(x) = 6x - 6x^2 = 6x(1-x) \rightarrow$ zeros: $x=0, x=1$.

$f''(x) = 6 - 12x = 6(1-2x) \rightarrow$ zeros: $x = \frac{1}{2}$

Write down a table with signs of f', f''

f''	+	+	-	-
f'	- ⊕	⊕ ⊕	⊕ ⊕	⊕ ⊖
f	DEC CU	INC CU	INC CB	DEC CB

f'' is continuous
 $f'(0) = 0$ & $f''(0) > 0 \Rightarrow 0$ is loc. MIN
 $f'(1) = 0$ & $f''(1) < 0 \Rightarrow 1$ is loc. MAX
 Alternatively: $f'(x) < 0$ for $x < 0$
 $f'(x) > 0$ for $x > 0$ } $x=0$ local MIN (close to 0)

Similarly $f' + \left| \begin{matrix} x=1 \\ - \end{matrix} \right.$ so $x=1$ is a local MAX.

• Change in concavity at $x = \frac{1}{2}$ is only inflection point!

Examples (last time):

① $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$ ant.

• $f''(x) = 0 \Rightarrow x = 0, \pi, 2\pi, \dots$
 $-\pi, -2\pi, \dots$

• $f'(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
 $-\frac{\pi}{2}, -\frac{3\pi}{2}, \dots$

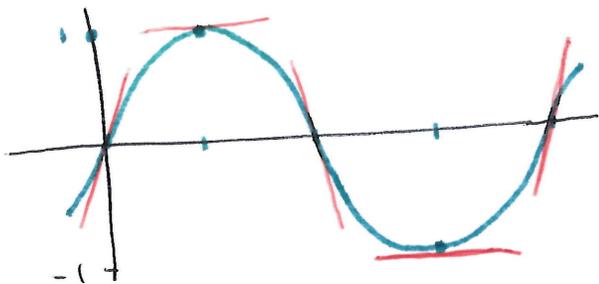
	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
f''	-	-	+	+	-
f'	+	-	-	+	+
f	DEC CB	DEC CB	DEC CU	DEC CU	INC CB

\downarrow L. MAX \downarrow inf pt \downarrow L. MIN \downarrow inf pt

2nd Deriv Test:

$f'(\frac{\pi}{2}) = 0$ $f''(\frac{\pi}{2}) < 0 \Rightarrow \frac{\pi}{2}$ local MAX

$f'(\frac{3\pi}{2}) = 0$ $f''(\frac{3\pi}{2}) > 0 \Rightarrow \frac{3\pi}{2}$ local MIN



f is periodic with period = 2π ($f(x+2\pi) = f(x)$)

② $f(t) = t^5 - 5t + 1 \Rightarrow f'(t) = 5t^4 - 5 = 5(t^2+1)(t+1)(t-1)$
 $f''(t) = 20t^3$

$f''(t) = 0 \Rightarrow t = 0$

$f'(t) = 0 \Rightarrow t = 1, -1$

$t = 0$ is the ONLY inf. pt

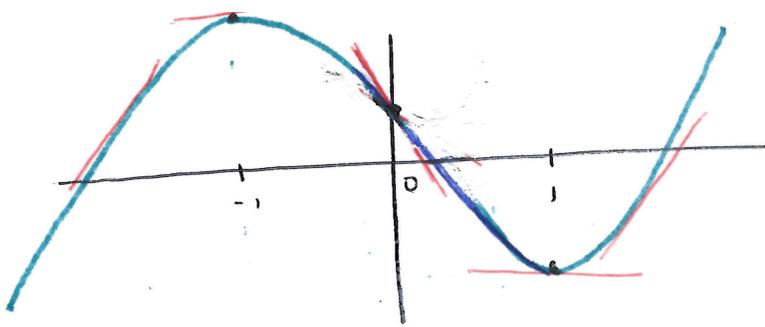
TEST $\Rightarrow t = -1$ is local MAX, $t = 1$ is local MIN.

f''	-	-	+	+
f'	⊕ ⊖	⊕ ⊖	⊖ ⊕	⊕ ⊕
f	INC CB	DEC CB	DEC CU	INC CU

$f(0) = 1$

$f(-1) = 5$

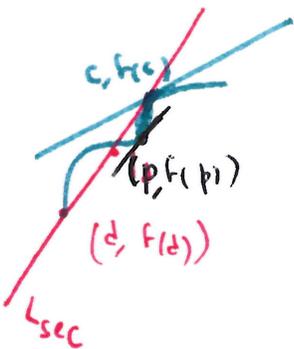
$f(1) = -3$



Proof of Concavity Test:

(1) Want to show the graph of f near $x=c$ lies above the tangent line at $(c, f(c))$. We argue by contradiction

Assume it fails so we have a point d arbitrarily close to c with $f(d)$ below the tangent line



Since f'' exists then f' is continuous, in particular it exists

so f is continuous on $[c, d]$

f is differentiable on (c, d)

By the Mean Value Theorem, we can find a point

p in (c, d) with $f'(p) = \frac{f(d) - f(c)}{d - c}$

Assume $d < c$, as in the picture

But $\frac{f(d) - f(c)}{d - c} = \text{slope of secant} > \text{slope of tangent at } (c, f(c)) = f'(c)$

So $f'(p) > f'(c)$ with $p < c$. This contradicts the fact that f' is increasing (because $f'' > 0$)

This proved that $f(d) \geq f(c)$, so the function is CU near any c .

We conclude f is CU on $[a, b]$.

(2) The proof for (2) uses the same arguments. \square

If $d > c$, then $\frac{f(d) - f(c)}{d - c} = \text{slope of secant} < \text{slope of tangent at } (c, f(c)) = f'(c)$

$$f'(p) = \frac{f(d) - f(c)}{d - c}$$

so $p > c$ & $f'(p) < f'(c)$. This contradicts the fact that f' is increasing! Again, we conclude that the function is CU near any c , so f is CU on $[a, b]$.