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Lecture XV : § 43 Applied Maximum & Minimum Problems  
 3.4.4 More Max-Min problems. Reflection & Refraction

Message of today's lecture: The solution of many applied problems revolves around maximizing or minimizing something.

[Challenge: Translate to a concrete function to maximize/minimize.]

Example: In physical sciences, nature often wants to minimize something

(1) energy used (Hamilton's Principle of Least Action)

(2) Time to travel from A to B (Fermat's Principle of Least Time)

and so the solution of a physical problem is often a MINIMIZATION problem

Example 2: In business, the simplest paradigm is:

(1) minimize cost

(2) maximize profit

KEY steps: Modeling = Find the equations that govern the problem

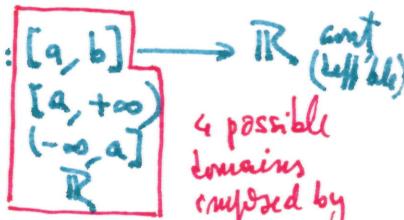
- Determine the constraints that have to be met

Easy part: Calculus of maximizing/minimizing  $f: [a, b] \rightarrow \mathbb{R}$  cont. (left side)

(1) Find the critical points:  $f'(x)=0$  or  $f'(x)$  does not exist at  $x$ .

(2) Compute the critical values:  $f(x)$  for  $x$  a crit. pt.

(3) Compute  $f$  at the endpoints of  $D_m f$  (unless they are  $\pm\infty$ )



→ If  $D_m f$  is bounded =  $[a, b]$  &  $f$  is continuous, we know  $f$  has extremal values, so we obtain it by comparing the critical values with  $f(a)$  &  $f(b)$ .

→ If  $D_m f$  is NOT bounded ( $\text{so } = [a, +\infty), (-\infty, a], \text{ or } \mathbb{R}$ ) we need to study  $\lim_{x \rightarrow +\infty} f(x)$  and/or  $\lim_{x \rightarrow -\infty} f(x)$ , since the function may not have max & min values. (it will if the limits =  $\pm\infty$ )

Eg:  $f(x) = x^2$   $f: [0, +\infty) \rightarrow \mathbb{R}$  has a min, but no max.

$f(x) = x^3$   $f: \mathbb{R} \rightarrow \mathbb{R}$  has no max & no min

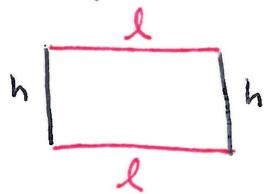
$f: (-\infty, a] \rightarrow \mathbb{R}$  has a max but no min

$f: [a, +\infty) \rightarrow \mathbb{R}$  " " min " " max

## Strategy for modeling:

1. Understand the "word problem"
2. Make a careful sketch if geometry is involve. Don't include any unwanted assumptions (e.g., symmetries, special features) that aren't there
3. Label the figure with the given data & suggestive name variables.  
Write down any constraints
4. Write down the equations involved & what needs to be maximized or minimized  
Transform the question into one involving a function of a single variable (e.g. time)
5. Solve the max/min problem & make sure the answer makes sense in the context of the problem.

Example 1 Show that the rectangle with maximum area for a fixed perimeter is a square.



- (1) Area =  $l \cdot h$   $\rightarrow$  Want to maximize Area
- (2) Perimeter =  $2l + 2h = 2(l+h) = P$  fixed  $\geq 0$
- CONSTRAINTS:  $l, h \geq 0$ .

Use (2) to turn Area into a single variable expression:

Solve<sup>(2)</sup> for  $l$ : 
$$l = \frac{P}{2} - h$$
 & replace in Area.

$$\text{Area}_{(h)} = \left(\frac{P}{2} - h\right)h = \frac{P}{2}h - h^2 \quad \text{for } h \geq 0$$

$$l = \frac{P}{2} - h \geq 0, \text{ so } \frac{P}{2} \geq h$$



Conclusion: Maximize  $A(h) = \frac{P}{2}h - h^2$  for  $0 \leq h \leq \frac{P}{2}$

Since  $A$  is continuous & we have  $A: [0, \frac{P}{2}] \rightarrow \mathbb{R}$  we know by the Extreme Values Thm that  $A$  has a max value. We just need to find it!

1. Crit points of  $A$ :  $A' = \frac{P}{2} - 2h = 0 \Rightarrow h = \frac{P}{4}$ .

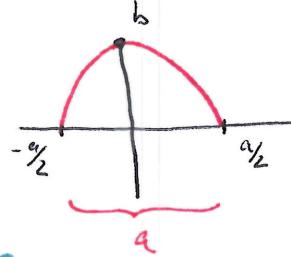
2. Crit Values:  $A\left(\frac{P}{4}\right) = \frac{P}{2}\frac{P}{4} - \frac{P^2}{16} = \frac{P^2}{8} - \frac{P^2}{16} = \frac{P^2}{16}$ .

3. Extremes of  $[0, \frac{P}{2}]$ :  $A(0) = 0$      $A\left(\frac{P}{2}\right) = 0$

4. Comparison gives  $A\left(\frac{P}{4}\right) = \frac{P^2}{16}$  as the winner,  $h = \frac{P}{4}$ , so 
$$l = \frac{P}{2} - \frac{P}{4} = \frac{P}{4} = h$$

Example 2: Given 2 positive constants  $a, b$  & consider the region between the parabola:  $a^2 y = a^2 b - 4b x^2$  and the  $x$ -axis. It is a parabolic segment of base  $a$  & height  $b$

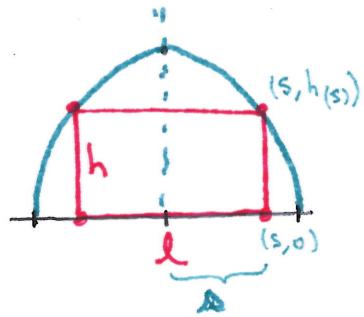
x-intercepts:  $y = 0 \Rightarrow a^2 b = 4b x^2 \Rightarrow a^2 = 4x^2 \Rightarrow x = \pm \frac{a}{2}$



Vertex of  
 $y = b - \frac{4b}{a^2} x^2$   
 $y' = -\frac{8b}{a^2} x = 0$

Find the base & height of the largest rectangle with lower base on the  $x$ -axis & upper vertex on the parabola.

Sln:



The problem is symmetric with respect to the  $y$ -axis

$$l = 2s$$

$$\text{Largest} = \text{largest in Area} = 2s \cdot h = 2s h(s)$$

$$h(s) = \frac{a^2 b - 4b s^2}{a^2} = b - \frac{4b}{a^2} s^2$$

$$\text{So } A(s) = 2s \left( b - \frac{4b}{a^2} s^2 \right) = 2bs - \frac{8b}{a^2} s^3$$

Constraints:  $0 \leq s \leq \frac{a}{2}$  Again a bound problem with a known to exist answer

- Crit pts:  $A'(s) = 2b - \frac{8b}{a^2} 3s^2 = 2b \left( 1 - \frac{12}{a^2} s^2 \right) = 0 \Rightarrow s^2 = \frac{a^2}{12}$

$$s = \frac{\pm a}{2\sqrt{3}} \quad \text{but} \quad s \geq 0 \quad \text{so we can discard the - sign soln.}$$

- Crit Value:  $A\left(\frac{a}{2\sqrt{3}}\right) = \frac{2a}{2\sqrt{3}} \left( b - \frac{4b}{a^2} \frac{a^2}{3} \right) = \frac{ab}{3\sqrt{3}}$

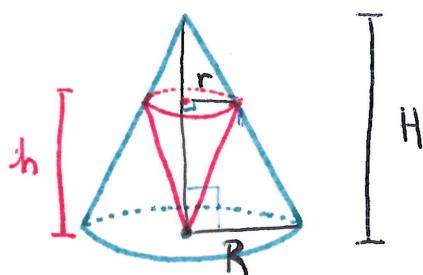
- End points:  $A(0) = 0$ ,  $A\left(\frac{a}{2}\right) = \frac{2a}{2} \left( b - \frac{4b}{a^2} \frac{a^2}{4} \right) = 0$

- Comparison:  $\frac{a}{2\sqrt{3}}$  winner, so base =  $\boxed{\frac{a}{\sqrt{3}}}$

$$\text{height} = b - \frac{4b}{a^2} \frac{a^2}{12} = b - \frac{b}{3} = \boxed{\frac{2b}{3}}$$

Example 3: A cone with height  $h$  is inscribed in a larger cone with height  $H$  so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when  $h = \frac{1}{3}H$ .

Soln:



Inscribed = inside + touching the boundary

$$\text{Vol Cone} = \frac{\pi R^2}{3} H$$

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. Big cone: height  $H \geq 0$  } fixed  
radius  $R \geq 0$

. Small cone: height  $h \geq 0$  Also:  $h \leq H$   
radius  $r \geq 0$   $r \leq R$

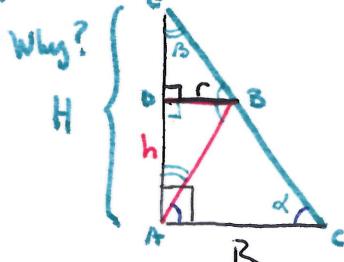
Constraints

$$0 \leq h \leq H$$

$$0 \leq r \leq R$$

GOAL: Maximize Volume subject to the condition that cone is inscribed in Cone.

Q: How to test inscription? Similarity of triangles  $\triangle D \sim \triangle A$



$$AB = BC \Rightarrow \hat{BAC} = \hat{ACB} (= \alpha)$$

$$DB \parallel AC \text{ so } \hat{DBE} = \alpha$$

$$\hat{DEB} = 90 - \alpha = \hat{DAB} (= \beta)$$

$$\text{So } \hat{ADB} = 90 - \beta = \alpha \text{ (from } \hat{ABD})$$

We conclude:  $\triangle ADB \sim \triangle ACE$  are similar (right) triangles (they have the same angles!)

$$\text{So } \frac{r}{h} = \frac{BB}{AD} = \text{sin } \beta = \frac{AC}{AE} = \frac{R}{H}$$

→ Also:  $\triangle ACE \sim \triangle DBE$  are similar triangles ( $DB \parallel AC$ )

$$\text{So } \frac{r}{R} = \frac{H-h}{H} \text{ gives } r = \frac{R}{H}(H-h)$$

Replace this in  $\text{Vol cone} = \pi \frac{R^2}{3} (H-h)^2 h$  cut on  $[0, H]$

$$\begin{aligned} 1. \text{ Crit pts: } V'(h) &= \frac{\pi R^2}{H^2} (2(H-h)(-1)h + (H-h)^2) = \frac{\pi R^2}{H^2} (H-h)(H-3h) \\ &= \frac{\pi R^2}{H^2} (H-h)(H-3h) \Rightarrow h=H \text{ or } h=\frac{H}{3} \end{aligned}$$

$$2. \text{ Crit Values: } V(H) = 0 \quad V\left(\frac{H}{3}\right) = \pi \frac{R^2}{H^2} \left(\frac{2H}{3}\right)^2 \frac{H}{3} = \frac{4\pi R^2 H}{27}$$

$$3. \text{ End pts: } V(0) = V(H) = 0 \quad \text{so max value for } \boxed{h = \frac{H}{3}}$$