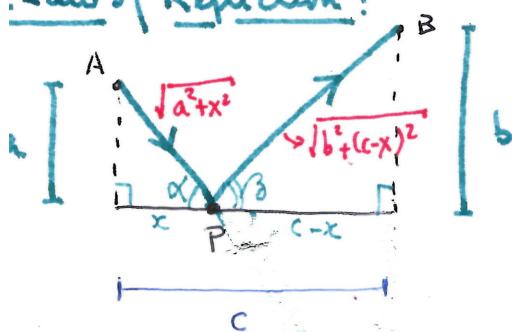


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Lecture XVI : § 4.4. Reflection & Refraction  
§ 4.5. Related Rates

Law of Reflection:



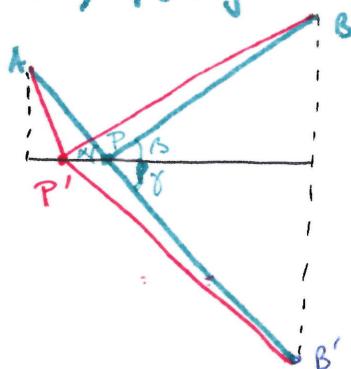
Ray of light travels from A to P, bounces on the mirror surface to reach B in the shortest amount of time, then  $\alpha = \beta$ .

Proof 1: Take  $B'$  = mirror image of B & draw P as the intersection of the line  $AB'$  w/ mirror.

Then  $\alpha = \gamma$  &  $\gamma = \beta$  shows  $\alpha = \beta$ .

Total length =  $AP + PB = AP + PB' = AB'$ .

Now, for any other point  $P'$  in the mirror, draw the same picture:



$$\text{Length} = AP' + P'B = AP' + P'B'$$

But  $P'B' + AP' > AB'$  (Why? Straight line is shortest path between 2 pts)

So the shortest path is indeed through P. &  $\alpha = \beta$   $\square$

in a single medium

Prof 2:  $L(x)$  needs to be minimize (light travels at constant speed  $v_0$ )  
to minimize time if we minimize distance traveled)

$L$  is continuous & differentiable (because  $a, b > 0$ )  
so the min must come from a critical point!

$$L'(x) = \frac{2x}{2\sqrt{a^2+x^2}} + \frac{-2(c-x)}{2\sqrt{b^2+(c-x)^2}} = \frac{x}{\sqrt{a^2+x^2}} - \frac{(c-x)}{\sqrt{b^2+(c-x)^2}}$$

$$L'(x) = 0 \text{ gives } \frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{b^2+(c-x)^2}}$$

$$\text{We square this: } \frac{x^2}{a^2+x^2} = \frac{(c-x)^2}{b^2+(c-x)^2} \text{ and } x^2(b^2+(c-x)^2) = (c-x)^2(a^2+x^2)$$

$$\text{& invert: } \frac{a^2+x^2}{x^2} = \frac{b^2+(c-x)^2}{(c-x)^2} \text{ so } \frac{a^2}{x^2} + 1 = \frac{b^2}{(c-x)^2} + 1$$

$$\text{so } \frac{(c-x)^2}{b^2} = \frac{x^2}{a^2} \text{ gives the quadratic equation in } x \text{ which we can solve. No need!}$$

$$\text{Note: } \tan \alpha = \frac{a}{x} = \frac{b}{c-x} = \tan \beta \quad \text{as } \alpha, \beta \in [0, \frac{\pi}{2}] \text{ so } \boxed{\alpha = \beta}$$

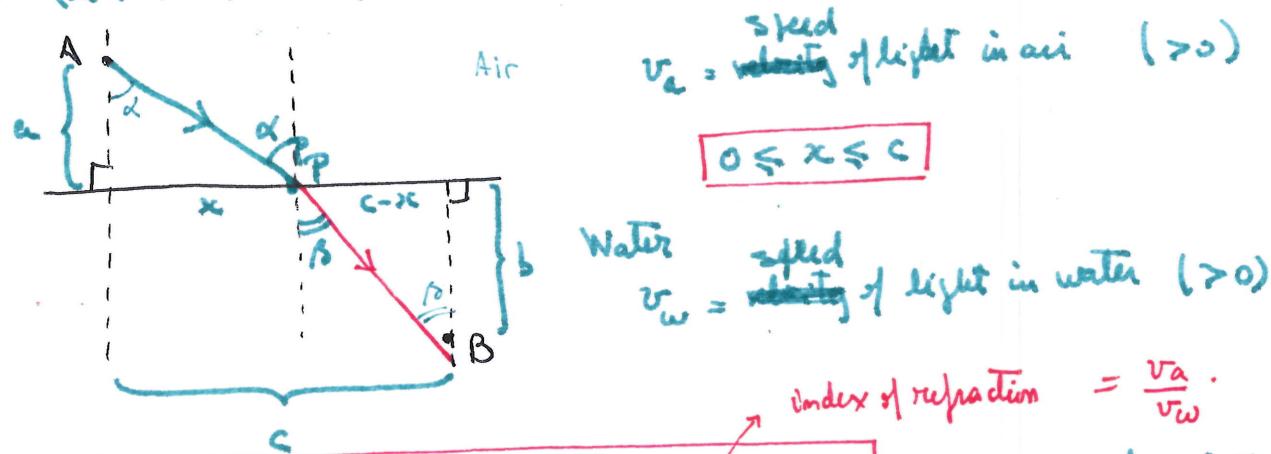
To check minimization, we can use The 2<sup>nd</sup> Derivative Test

$$L'' = \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{b^2}{(b^2+(c-x)^2)^{3/2}} > 0 \quad (\text{sum of positive numbers})$$

## § 2 Law of Refraction:

Q: What happens when the ray of light travels through 2 media (air & water / glass, ...)?

Again, we want to minimize the time it takes to travel from A (in one medium) to B (in a different medium)



**Snell's Law:**  $\frac{\sin(\alpha)}{\sin(\beta)} = \text{constant}$  → dependent ONLY on  $v_a$  &  $v_w$  & independent of the position of A & B.

Proof: Set this as a problem of minimizing the distance ~~the time it takes to travel~~:

$$\text{Time in the air} = T_{\text{air}} = \frac{\text{dist}}{\text{velocity}} = \frac{AP}{v_a} = \frac{\sqrt{a^2+x^2}}{v_a}$$

$$\text{Time in the water} = T_{\text{water}} = \frac{\text{dist}}{\text{velocity}} = \frac{PB}{v_w} = \frac{\sqrt{b^2+(c-x)^2}}{v_w}$$

$$\begin{aligned} \text{Total Time } T(x) &= T_{\text{air}} + T_{\text{water}} \\ &= \frac{1}{v_a} (a^2+x^2)^{1/2} + \frac{1}{v_w} (b^2+(c-x)^2)^{1/2} \end{aligned}$$

(Reflection for  $v_a = v_w$ )

As with the Reflection, the minimum value will not be realized for  $x=0$  or  $x=c$ . But we need to check this, <sup>e.g.</sup> by comparing with cut values,

$$T'(x) = \frac{1}{v_a} \frac{x}{\sqrt{a^2+x^2}} - \frac{1}{v_w} \frac{(c-x)}{\sqrt{b^2+(c-x)^2}} = \frac{1}{v_a} \frac{\sin \alpha}{\sin \beta} - \frac{\sin \beta}{v_w}$$

$$T'(x)=0 \text{ gives } \frac{\sin \alpha}{v_a} = \frac{\sin \beta}{v_w} \quad \text{so} \quad \boxed{\frac{\sin \alpha}{\sin \beta} = \frac{v_a}{v_w}}$$

To check for min / max / pt of inflection, use the 2<sup>nd</sup> Derivatives Test.

$$T''(x) = \frac{1}{r_a} \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{1}{r_w} \frac{b^2}{(b^2+(c-x)^2)^{3/2}} > 0 .$$

Note  $T'(0) = -\frac{c}{r_w b} < 0$  so  $T$  is str decreasing near 0.

$T'(c) = \frac{c}{r_a \sqrt{a^2+c^2}} > 0$  .  $T$  is str increasing near c

This shows that  $x=0$  &  $x=c$  cannot give minimum value. This is because we know that the min value  $x$  must be a critical point & so Snell's law is verified (we have a solution to  $T(x) = 0$ , namely the min value!)

### § 3 Related Rates:

- We have 2 or 3 quantities linked together by a constraint
- The system is changing ("in motion") and so all quantities are changing ("with time")

GOAL: Compute the rate of change of one quantity in terms of the known rate of change of the remaining quantities.

Tools: • implicit differentiation  
• chain rule + substitution of known values.

Only difficulty = modelling the problem

• Same strategy as w/ max-min problems

1. Draw diagram if appropriate
2. Label figures but DON'T fix quantities that are changing
3. Novelty = Find the relationship between the varying quantities  
• Identify the data & the unknown rate. Look at units of measurement

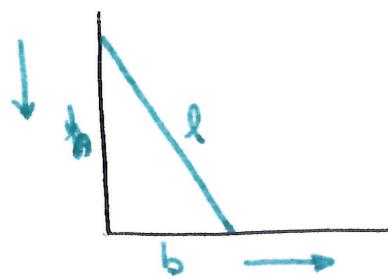
Examples: ① Air being pumped into a spherical balloon whose volume increases at a given rate. Find the rate of growth of the radius of the balloon.

Soln:  $V(r) = \frac{4}{3}\pi r^3$       Know  $\frac{dV}{dt} = \frac{3}{3}\pi r^2 r'$        $r(1) = 8$   
chain Rule

Want to solve for  $r'$ ,  $r' = \frac{8}{4\pi r^2} = \frac{2}{\pi r^2} = \frac{2}{\pi (\frac{3}{4}\pi V)^{2/3}}$

At any given time, we know  $r(1) = \sqrt[3]{\frac{3}{4\pi} V(1)}$ , so we can compute  $r'(1)$ .

② Speed at which a ladder slides down a wall if it slides away from the wall at a fixed rate



$l$  = length of the ladder is fixed = (eg 13 ft)

Equation 
$$l^2 = b^2 + h^2$$

Know  $b'(t)$  (eg 6 ft/min) Want to complete:  $h'(t)$  (when  $b(t) = 5 \text{ ft}$ )

Use implicit differentiation :  $D = 2b \frac{b'}{b} + 2h \frac{h'}{h}$   
 & solve for  $h'(t)$

$$h'(t) = \frac{-b(t)b'(t)}{h(t)} = -\frac{5 \cdot 6}{\sqrt{13^2 - 5^2}} = -\frac{30}{12} = \boxed{-\frac{5}{2} \text{ ft/min}}$$

③ If a snowball melts so that its surface area decreases at a rate of  $1 \text{ cm}^2/\text{min}$ , find the rate at which the diameter decreases when the diameter is  $10 \text{ cm}$ .

Quantities :  $A(t)$  = surface area

•  $d(t)$  = diameter of the snowball

•  $t$  = time

Standard assumption = snowball is a sphere.

Formula for Area :  $A = 4\pi r^2$

$$A = 4\pi \left(\frac{d}{2}\right)^2 = \pi d^2$$

Known data :  $d(t_0) = 10 \text{ cm}$

•  $\frac{dA}{dt}(t_0) = -1 \text{ cm}^2/\text{min}$  *decreases*

Unknown :  $d'(t)$ .

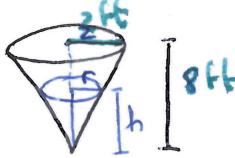
Use implicit differentiation :  $-1 = A'(t) = \pi 2d \frac{d'}{dt}(t_0) = 2\pi 10 d'(t_0)$

Then 
$$d'(t) = \boxed{-\frac{1}{20\pi} \text{ cm/min}}$$

④ A conical <sup>water</sup> tank with its vertex down is  $8 \text{ ft}$  high &  $4 \text{ ft}$  in diameter at the top. The tank is full & water leaks through a hole in the bottom at a rate of  $1 \text{ ft}^3/\text{min}$ . Find the rate at which the water level is falling when the tank is  $\frac{7}{8}$  empty.

• Vol =  $\frac{1}{3}\pi r^2 h$

• Full Vol =  $\frac{1}{3}\pi 4^2 \cdot 8 = \frac{32}{3}\pi \text{ ft}^3$



Unknown :  $h'(t)$

Constraint :  $h = \text{height} = \frac{1}{3} \text{ Total ht} = 1 \text{ ft}$

$$\frac{r(t)}{h(t)} = \frac{2}{8} \Rightarrow \boxed{r(t) = \frac{2}{8} h(t)}$$

Known relation :  $r(t) = \frac{1}{4} h(t)$   $\Rightarrow$  replace in Vol

$$V(t) = \frac{1}{3} \pi \frac{1}{16} h(t)^2 h'(t) = \frac{\pi}{48} h^3(t)$$

Use implicit differentiation :  $\frac{-1 \text{ ft}}{\text{min}} = \frac{dV}{dt} = \frac{3\pi}{48} h^2(t) h'(t)$

$$\text{so } -1 = \frac{\pi}{16} \cdot 1 \cdot h'(t) \quad \text{yires} \quad \boxed{h'(t) = -\frac{16}{\pi} \text{ ft/min}}$$