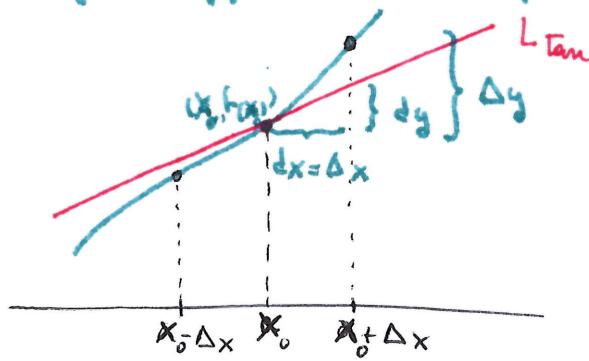


Lecture XVII: 55.2 Differentials and Tangent Line Approximations

§1. Linear approximation & differentials

Idea: Near a given point, the tangent line to a graph at a given point is a good approximation of the graph.



increase in the graph = $(\Delta x, \Delta y)$

where $\Delta y = f(x_0 + \Delta x) - f(x_0)$

increase in the tangent = (dx, dy)

$$y = f'(x_0)(x - x_0) + f(x_0) \rightarrow \text{tangent line}$$

$$\boxed{\begin{aligned} dy &= y - f(x_0) = f'(x_0) dx \\ dx &= x - x_0 = \Delta x \end{aligned}} \quad (*)$$

This notation of differentials dy & dx is inspired by Leibniz:

$$y = f(x) \rightsquigarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad \text{"as if"} \quad \lim_{\Delta x \rightarrow 0} \Delta y = dy, \quad \lim_{\Delta x \rightarrow 0} \Delta x = dx$$

Of course, both limits are 0. But there is some sense we can make of dx & dy so that $f'(x) = \frac{dy}{dx}$ has meaning?

A: Yes. This is the notion of a differential, as defined in (*).

The differential dx is an independent variable, and the differential dy is defined in terms of dx by $dy = f'(x_0) dx$. $\therefore df$

Summary: $\Delta y = \text{change along the graph of the function } f \text{ (= a curve) as } x \rightarrow x + \Delta x$
 $dy = \text{_____ tangent line as } x \rightarrow x + dx$.

But: If f is linear, then $\Delta y = dy = m dx$ so both $\frac{\Delta y}{\Delta x} = m = \frac{dy}{dx}$

$$\text{Eg: } f(x) = x^2 \rightsquigarrow f'(x) = 2x \text{ so } dy = 2x dx$$

$$f(x) = \sin(x) \rightsquigarrow f'(x) = \cos(x) \text{ so } dy = \cos(x) dx$$

$$\text{Equivalently } df = 2x dx \text{ & } df = \cos(x) dx.$$

§2 Differentiation Formulas in Differential Notation

(1) Power rule: $y = u^n \rightsquigarrow dy = n u^{n-1} du$

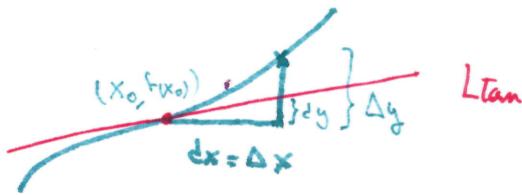
(2) Product Rule $y = u \cdot v \rightsquigarrow dy = d(uv) = u dv + v du$
 [why? $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ & multiply by dx]

$$(3) \text{ Quotient Rule: } d\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}.$$

$$(4) \text{ Chain Rule: } y = f(u) \quad u = g(x)$$

$$dy = f'(u) du \quad \& \quad du = g'(x) dx \quad \text{so} \quad dy = f'(u)g'(x)dx$$

§3 Tangent line approximations



Near the point of tangency, the graph of f is very close to the tangent line. As $\Delta x = dx$ becomes small, then the secant line \rightarrow tang. line
 $\Delta y \rightarrow dy$

Conclusion: For small $\Delta x \approx dx$, the change in the tangent line is a good approximation to the change in $f(x)$.

$$\Delta y = f(x+\Delta x) - f(x) \approx dy$$

$$\text{Equivalently: } f(x+\Delta x) \approx f(x) + dy$$

$$f(x+\Delta x) \approx f(x) + \underbrace{f'(x)dx}$$

The linear function $L(x) = f(x_0) + f'(x_0)(x-x_0)$ is the linearization of f at x_0 . We use $L(x)$ to approximate $f(x)$ near x_0 .

Application: Find approximations of square roots, cube roots, etc.

Example 1: Find the linear approximation of $f(x) = \sqrt{1-x}$ at $a=0$ & estimate

$$f'(x) = \frac{1}{\sqrt{1-x}}(-1) \Rightarrow f'(0) = -1, \quad f(0) = \sqrt{1} = 1 \quad \sqrt{0.9}, \sqrt{0.99}.$$

$$L(x) = 1 + (-1)(x-0) = 1-x$$

$f(0.1) = \sqrt{0.9}$ & $f(0.01) = \sqrt{0.99}$ are approximated by

$$L(0.1) = 1-0.1 = 0.9 \quad \& \quad L(0.01) = 0.99, \text{ respectively}$$

Example 2: Find the linear approximation of $\sin(x)$ at $x=0$

$$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1 \Rightarrow L(x) = \sin(0) + \frac{1}{0}(x-0) = x$$

So x is a good approximation of $\sin x$ near 0.

Notice: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ so $\sin x$ behaves like x near $x=0$.

Example 3 $\sqrt[3]{28}$ $3^3 = 27 \rightarrow \sqrt[3]{28} = \sqrt[3]{27+1}$ use $f(x) = \sqrt[3]{x}$ (3)

$$f'(x) = \frac{1}{3}(x)^{-\frac{2}{3}} \quad f'(27) = \frac{1}{3}27^{-\frac{2}{3}} = \frac{1}{3 \cdot 9} = \frac{1}{27}, \quad f(27) = 3.$$

$$L(x) = 3 + \frac{1}{27}(x-27) = 2 + \frac{x}{27} \quad \text{so } \sqrt[3]{28} \approx 3 + \frac{1}{27} \approx 3.037$$

Example 4 The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk & the relative error/percentage error

$$\text{Area } A(r) = \pi r^2 \Rightarrow dA = 2\pi r dr \quad \boxed{\text{Error} = dr}$$

$$\text{When } r = 24, dr = 0.2 \Rightarrow dA = 2\pi r dr \approx 2\pi \cdot 24 \cdot 0.2 = \boxed{9.6\pi}$$

$$\text{Maximum error} = 9.6\pi$$

$$\text{Relative error} \approx \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{9.6\pi}{\pi (24)^2} = \boxed{\frac{1}{60}} \quad ; \quad \frac{\Delta r}{r} \approx \frac{dr}{r} = \frac{0.2}{24}$$

$$\text{Relative error: } \frac{\Delta A}{A} = \boxed{2} \frac{\Delta r}{r} \quad \text{shows the relative error} = \boxed{2} = \boxed{\frac{1}{100}}.$$

Example 5: If the earth's radius increased by 1 ft, how much would the surface area increase?

$$A = 4\pi r^2 \quad \text{Radius of earth} \approx 4,000 \text{ mi} \quad , \quad dr = 1 \text{ ft} = ? \text{ mi}$$

$$\Delta A \approx dA = 8\pi r dr = 8\pi \frac{4000}{5280} \approx \boxed{19.04 (\text{mi})^2} = \boxed{\frac{1}{5280} \text{ mi}}$$