

§ 1 Antiderivatives

Up to now, we've discuss rules to find f' given a function f .

IDEA: We want to reverse the process ("antiderivatives or antiderivatives" meaning: given $f(x)$, can we find $F(x)$ with $\frac{dF}{dx} = f$?

Example: $\frac{dF}{dx} = 3x^2 \Rightarrow F = x^3$, but $x^3 + 1$ (so $\frac{dF}{dx} = f \text{ also works}$)

$\frac{dF}{dx} = \cos(x) \Rightarrow F = \sin(x)$, but $\sin(x) - \pi$ also works

Guiding question: Can we recover a function from its derivative?

Proposition: If $F(x)$ & $G(x)$ are two functions with the same derivative $f(x)$ on a certain interval $[a, b]$, then $F(x) - G(x)$ is constant in (a, b)

Proof: $\frac{d}{dx}(F(x) - G(x)) = \frac{dF}{dx} - \frac{dG}{dx} = f(x) - f(x) = 0 \quad \& \quad F(x) - G(x)$

is differentiable with derivative $\stackrel{\text{Additive Rule}}{=} 0$. By MVT (Lecture VII), we conclude the function $F(x) - G(x)$ is constant in (a, b) , so $F(x) = G(x) + c$ for a fixed constant c . \blacksquare

Conclusion: If $F(x)$ is an antiderivative of a function f on an interval, all others are of the form $F(x) + C$ where C is an arbitrary constant.

Notation: $\int f(x) dx = F(x) + C$ (indefinite integral / antiderivative)

Remark: Every formula for a derivative gives a corresponding formula for an indefinite integral.

① Power Rule: $\frac{d}{dx} x^n = n x^{n-1} \Rightarrow dx^n = n x^{n-1} dx ; x^{n-1} dx = \frac{dx^n}{n}$

Then $\int n x^{n-1} dx = x^n + C$, $\int_{(m=n-1)} x^{m+n} dx = \frac{x^{m+1}}{m+1} + C$ $\text{for } m \neq -1$

② Trig Rule: $\int \sin(x) dx = -\cos(x) + C$ $\int \cos(x) dx = \sin(x) + C$

$\int \sec^2(x) dx = \tan(x) + C$, $\int \csc^2(x) dx = -\cot(x)$

$\int \operatorname{tanh}^2(x) \sec(x) dx = \sec(x) + C$, $\int \operatorname{wt}^2(x) (\sec(x)) = -\csc(x) + C$

③ More general rules: multiplication by constant, addition.

• If c is a constant, $\frac{d}{dx}(cf(x)) = c \frac{df}{dx}$

so $\int cf(x) dx = c \int f(x) dx$

• $d(F(x) + G(x)) = F'(x) dx + G'(x) dx = f(x) dx + g(x) dx$

so $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$.

• Let's get antiderivatives of polynomials!

Eg $\int 3x^2 + 4x + 2 dx = 3 \int x^2 dx + 4 \int x dx + 2 \int 1 dx$
 $= x^3 + C_1 + 4 \underbrace{\left(\frac{x^2}{2} + C_2\right)}_{= C \text{ arbitrary const.}} + 2x + C_3$
 $= x^3 + 4 \frac{x^2}{2} + 2x + \underbrace{(C_1 + 4C_2 + C_3)}_{= C \text{ arbitrary const.}}$

• We can integrate any power except $\frac{1}{x}$.

Eg $\int x^{1/3} (x+2)^2 dx = \int x^{1/3} (x^2 + 4x + 4) dx = \int x^{1/3} + 4x^{4/3} + 4x^{1/3} dx$
 $= \frac{x^{10/3}}{10/3} + 4 \frac{x^{7/3}}{7/3} + 4 \frac{x^{4/3}}{4/3} + C = \frac{3}{10} x^{10/3} + \frac{12}{7} x^{7/3} + 3x^{4/3} + C$

④ Most subtle technique = Chain Rule via Substitution

§ 2 Substitution:

Idea: $df = f(u) du = u^{1/2} du$ then $\int u^{1/2} du = \frac{2}{3} u^{3/2} + C$

What if $u = u(x) = x^2 + 1$, Then $du = u' dx = 2x dx$

So $df = f(u) du = \sqrt{x^2+1} 2x dx = f(u) u'(x) dx$

$$\int \sqrt{x^2+1} 2x dx = \int f(u) \underbrace{u'(x) dx}_{du} = \int f(u) du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (x^2+1)^{3/2} + C$$

We can check the result by computing the derivative: $\stackrel{\text{Substitute } u=x^2+1}{\frac{d}{dx} \left(\frac{2}{3} (x^2+1)^{3/2} + C \right)}$

$$\stackrel{\text{Chain Rule}}{\uparrow} = \frac{2}{3} \cdot \frac{3}{2} (x^2+1)^{3/2-1} \cdot 2x = \sqrt{x^2+1} 2x.$$

From this example we can derive the general rule:

Substitution Rule: If $F = f(g(x))$, write $\bar{F} = f(u)$ & $u = g(x)$.
 then $d\bar{F} = f'(u) du$ & $du = g'(x) dx$, so $d\bar{F} = f'(g(x)) g'(x) dx$

then $\int d\bar{F} = \int f'(g(x)) \frac{g'(x)}{du} du = \int f'(u) du = f(u) + C$
 $= f(g(x)) + C$

Q: How do we use this?

If we want to solve $\int \sqrt{x^2+1} 2x dx$, we recognize $(x^2+1)' = 2x$.

so we set $u = x^2 + 1$, $du = 2x dx$

$$\int \sqrt{x^2+1} \frac{2x \, dx}{\cancel{2x}} = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = F(u)$$

But we always put back the result in terms of the original variables,
in our case x , so $\int \sqrt{x^2+1} \cdot 2x \, dx = F(x^2+1) = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$

Notes ① The integration involves substitution

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The check (verification) involves the chain rule.

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- ② The Substitution Rule is fairly straight forward if we view the differentiation step as computing a differential.

③ Finding $u = u(x)$ in the expression can be tricky! We must be able to write the integrand $f(x) dx$ as $g(u) du$, ie making sure no x appears!

$$\text{Example: } \textcircled{1} \int w^5(x) \sin x \, dx = \int u^5 (-du) = - \int u^5 du = -\frac{u^6}{6} + C$$

\downarrow

$$\begin{aligned} u &= w(x) \\ du &= -\sin(x) \end{aligned}$$

$$= -\frac{w^6(x)}{6} + C.$$

$$\textcircled{2} \quad \int \sin(5x) dx = \int \sin(u) \frac{du}{5} = \frac{1}{5} \int \sin(u) du = -\frac{\cos u}{5} + C \\ = -\frac{\cos(5x)}{5} + C$$

$u = 5x$
 $du = 5dx$

$$\begin{aligned} \textcircled{3} \quad \int \frac{dx}{(x-7)^7} &= \int \frac{1}{u^7} du = \int u^{-7} du = \frac{u^{-6}}{-6} + C = -\frac{(x-7)^{-6}}{6} + C \\ &\stackrel{u=x-7}{=} \frac{1}{6(x-7)^6} + C \\ &\stackrel{du=dx}{=} \end{aligned}$$