

Lecture XIX: §§4. Differential Equations. Separation of variables
 §§5: Motion under Gravity

§1. Differential Equations & separation of variables

The theory of diff'l equations aims to recover a function from a relation among its derivatives

Examples ① $y' = 2x$ \rightsquigarrow By integration: $dy = 2x dx \Rightarrow \int dy = \int 2x dx$

Then $y_{(x)} = x^2 + C$ for any constant

② $y'' = -y$ $\rightsquigarrow y'' + y = 0$ \rightsquigarrow Soln: $y_{(x)} = a \sin x + b \cos(x)$ "injection"

③ $y''' + x^2 y'' - y = 0 \rightsquigarrow$ Soln = ? $\rightsquigarrow a, b$ constants

Certain simple O.D.E. (ordinary diff'l eqns.) can be solve via separation of variables (analog of implicit differentiation)

Ex. $\frac{dy}{dx} = -x^2 y^2$ or $dy = -x^2 y^2 dx$

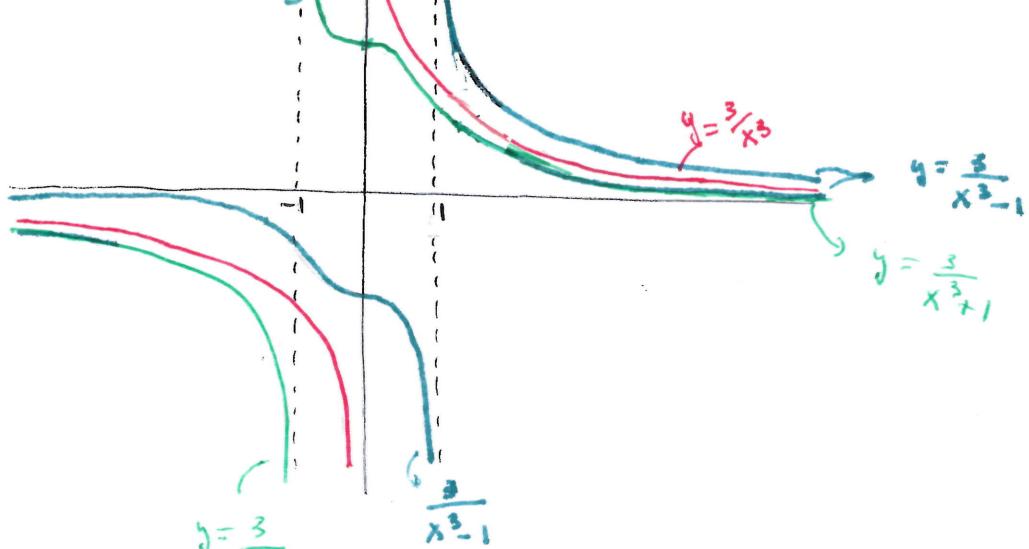
(1) We put different variables on different sides of the equation:

$$-\frac{dy}{y^2} = +x^2 dx$$

(2) Integrate both sides: $\frac{1}{y} + C_2 = \int -\frac{dy}{y^2} = \int +x^2 dx = \frac{x^3}{3} + C_1$

$$\text{so } y = \frac{1}{\frac{x^3}{3} + C} = \frac{3}{x^3 + C}$$

\rightsquigarrow general solution. { 1-parameter family of solutions: param = C,



Particular solution: choose a value for C

$$\rightarrow 3 \text{ solutions: } \frac{3}{x^3 + 1}, \frac{3}{x^3}, \frac{3}{x^3 - 1}$$

$$C = 0, 1, -1.$$

(3) We can uniquely determine C if we know an initial condition, that is, a value for $y(x_0)$ for some x_0 [eg: $y(0) = 3 \Rightarrow C = 1$]

Note: Separation of variables only works for very special O.D.E.s:

$F(y', y, x) = 0$ (1st order) which can be split into

$$g(y) dy = f(x) dx$$

Nu-example: $y' = \frac{x+y}{x-y}$

Ex. $\frac{dy}{dx} = 2y^2 (4x^3 + 4x^{-3}) \Rightarrow y(0)=1 \Rightarrow \frac{dy}{y^2} = (8x^3 + 8x^{-3}) dx$

$$\Rightarrow -y^{-1} = 2x^4 - 4x^{-2} + C$$

Soln: $y = \frac{1}{-2x^4 + 4x^{-2} + C}$

$$y(0) = \frac{1}{C} = 1 \Rightarrow C = 1$$

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3. Motion under gravity: Position = $s(t)$, Velocity = $s'(t)$,

Newton's law of motion:

$$t = \text{time } (t \geq 0)$$

$$\text{Velocity} = s'(t)$$

I. A particle in a state of rest or motion will continue to be so unless an external force is applied to it.

II. $a(t) = \text{acceleration} = s''(t)$ $a = \frac{F}{m}$. equin. $\boxed{F = m a}$
 $m = \text{mass}$ $(*)$ $\boxed{F = m s''(t)}$ \Rightarrow ODE.

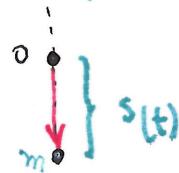
$F = \text{force}$

III. To every action, there is an equal and opposite reaction.

Given 2 initial conditions (eg $s(0)$, $v(0)$) we can get a particular solution to (*)

Example 1 Gravity force induces acceleration $\underset{\text{constant}}{g} \approx 9.8 \frac{\text{m}}{\text{s}^2} = 32 \frac{\text{ft}}{\text{s}^2}$

Find the motion of a stone of mass m which is dropped from a point above the surface of the earth.



Initial conditions: $s(0) = 0$
 $v(0) = 0$ (stone is dropped)
 \rightarrow (no air resistance)

Eqn $m a(t) = F = m \cdot g \Rightarrow a(t) = s''(t) = g$

Solve by integrating: $v(t) = s'(t) = \int g dt = g t + C_1$

But $v(0) = 0$ yields $C_1 = 0 \Rightarrow v(t) = g t$

Integrate again: $s(t) = \int v(t) dt = \int s'(t) dt = \int g t dt \Rightarrow s(t) = \frac{g t^2}{2} + C_2$

13)

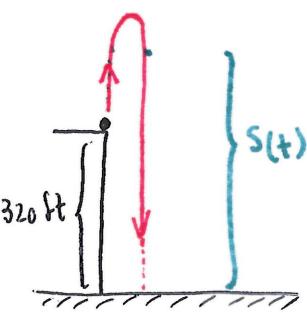
Use $s(0)=0$ to solve for C_2 : $0=s(0)=0+C_2$, so $C_2=0$

Solution :

$$s(t) = \frac{1}{2}gt^2$$

In general: $s(t) = \frac{1}{2}gt^2 + C_1t + C_2$ for C_1, C_2 constants, determined by 2 independent initial conditions.

Example 2: Assume a stone is thrown upwards at 128 ft/s from the roof of a building 320 ft high. Find the trajectory and determine its maximum height & at what time does the stone hit the ground.



Initial conditions : $s(0) = 320$

$$s'(0) = 128 \quad (\text{vec up so + sign})$$

$$F = -mg = s''(t) \cdot m \Rightarrow s''(t) = -g$$

$$\text{Integrate: } s'(t) = -gt + C_1$$

$$\text{One more time: } s(t) = \int g t + C_1 dt = -\frac{1}{2}gt^2 + C_1 t + C_2$$

$$\text{Use initial conditions: } \left. \begin{array}{l} s(0) = C_2 = 320 \\ s'(0) = C_1 = 128 \end{array} \right\} s(t) = -\frac{1}{2}gt^2 + 128t + 320.$$

$$\bullet \text{Maximal height: } s'(t) = -32t + 128 = 0 \Rightarrow t = \frac{128}{32} = 4 > 0$$

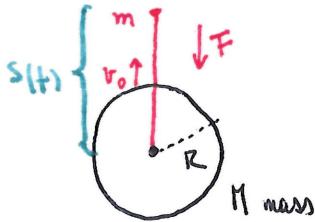
$$s(4) = -16 \cdot 4^2 + 128 \cdot 4 + 320 = 576 \text{ FT}$$

• When is $s(t) = 0$? \Rightarrow Solve using the quadratic eqn: $t = 10$ & $t = -2$. Since $t \geq 0$, we know we hit the ground after 10 sec.

$$\bullet \text{Speed? } |s'(10)| = |-32 \cdot 10 + 128| = |-192| = 192 \text{ ft/s}$$

§ 3. Escape Velocity:

GOAL: Determine the initial velocity v_{0w} which should vertically fire a rocket for it to come to rest & escape completely from the earth's gravitational attraction.



Newton's Law of gravitation: any 2 particles of matter attract each other with a force that is jointly proportional to their masses & inversely proportional to the square of the distance between them:

$$F = -G \frac{M \cdot m}{r^2}$$

$G > 0$ constant

• Can assume earth = particle of mass M located at its center.

We get $m \ddot{s}(t) = -\frac{GMm}{s^2(t)}$ so $s''(t) = -\frac{GM}{s^2(t)}$ (14)

Note At the ground: $s''(t) = -g$ when $s(t) = R$.

We get $-g = -\frac{GM}{R^2}$ so $-GM = -gR^2$.

We replace back in (14) to get $v'(t) = s''(t) = -\frac{gR^2}{s^2(t)}$ [Want to find v]

We rewrite this expression with the chain rule (thinking of s as the variable!)

$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = \frac{dv}{ds} \cdot v$$

We get $\frac{dv}{ds} \cdot v = -\frac{gR^2}{s^2}$ $\Rightarrow v dv = -\frac{gR^2}{s^2} ds$ separate variables!

We integrate both sides: $\frac{v^2}{2} = g \frac{R^2}{s} + C$

Initial velocity: $\frac{v_0^2}{2} = g \frac{R^2}{R} + C$ so $C = \frac{v_0^2}{2} - gR$

We get $\frac{v^2}{2} = g \frac{R^2}{s(t)} + (\frac{v_0^2}{2} - gR)$

To escape gravitational force, we need $v_{(t)} > 0$ for all t .

$\frac{gR^2}{s(t)} \xrightarrow[t \rightarrow \infty]{} 0^+$ if we escape earth

We can only guarantee we remain with $v(t) > 0$ for all t only if

$$\frac{v_0^2}{2} - gR \geq 0 \quad \text{that is} \quad v_0 \geq \sqrt{2gR} \quad (\text{know } v_0 > 0)$$

For the earth: $\sqrt{2gR} \approx 7 \text{ mi/s} = 25,000 \text{ mi/h}$

$$R = 4000 \text{ mi}$$

$$g = 32 \text{ ft/s}^2 = \frac{32}{5280} \text{ mi/s}^2$$

$\sqrt{2gR}$ escape velocity

$$= \sqrt{2 \frac{GM}{R^2}} = \sqrt{\frac{2GM}{R}}$$

Same formula works for any planet. (For different values of (R, g))

If the mass M is preserved but R decreases to R' , then $v_{(t)}$ increases (because $v_{(t)}$ is decreasing so $r(r') > r(R)$)



If escape velocity > speed of light, the light never escaped the orbit into black holes!