

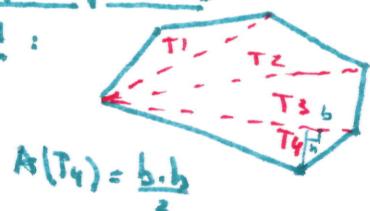
Lecture XX : § 6.2 The Problem of Areas

§ 6.3 The sigma notation & certain special sums

GOAL: Find a method to compute areas of regions enclosed by curves.

§1. The Problem of areas:

Example 1 :



$$A(T_4) = \frac{b \cdot h}{2}$$

closed polygon (not necessarily regular)

Triangulate it & add up the areas of the Δ s.

triangles = # sides - 2.

What about other, non linear, ^{convex} regions? Use method of exhaustion.

① Pick a finite number of points in the boundary: P_1, \dots, P_n (clockwise)

② Fix a point P in the interior

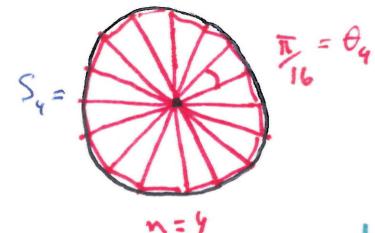
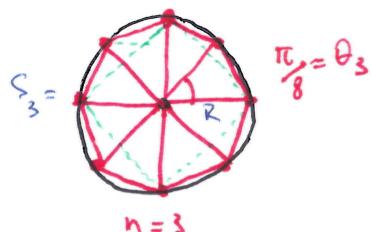
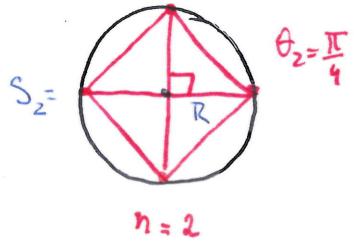
③ Construct all triangles $PP_1P_2, \dots, PP_{n-1}P_n, PP_nP_1$

Note since R is convex, all triangles are inside S

④ Make $n \rightarrow +\infty$ & the points P_1, \dots, P_n to be close to each other.

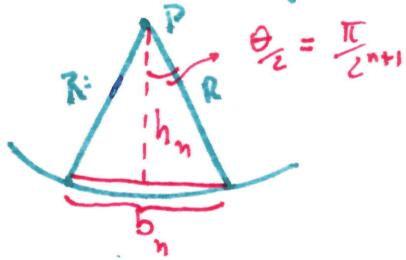
The triangles cover almost all of S , so they approximate $\text{Area}(S)$.

Example 2: S = circle of radius R , $P = (0,0)$, choose P_1, \dots, P_{2^n} as the vertices of a regular 2^n -gon.



Q: Why 2^n ? At each step we divide the angle θ by 2 \Rightarrow get $\theta = \frac{\pi}{2^n}$

- At each step: S_n to contained in S_{n+1} (square \subseteq octagon $\subseteq \dots$)
- Easy to compute $\text{Area} = 2^n \cdot \text{Area of 1 triangle}$



$$\begin{aligned} \text{Area}_{\text{Polyagon}} &= 2^n \cdot \text{Area}(\Delta) = 2^n \cdot \frac{h_n b_n}{2} \\ &= \frac{h_n}{2} \left(b_n + \dots + b_n \right) = \frac{h_n}{2} P_n \end{aligned}$$

where P_n = perimeter of the 2^n -gon!

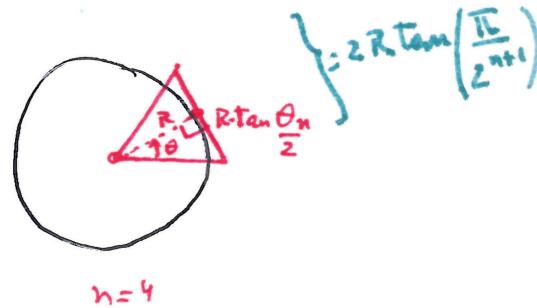
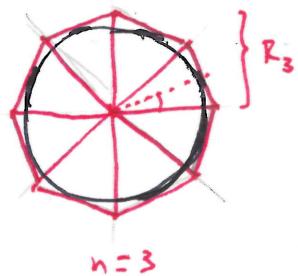
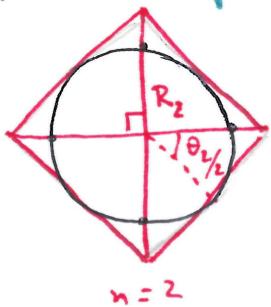
Q: What happens as we exhaust the circle? i.e., as $n \rightarrow +\infty$?

- $h_n \rightarrow R$ = radius of the circle

$P_n \rightarrow C$ = circumference of the circle $= 2\pi R$

$$\text{So } A_{\text{circle}} = \lim_{n \rightarrow \infty} (\text{Area polygon } S_n) = \frac{1}{2} R \cdot C = \boxed{\pi R^2} \quad \square$$

Q: What if we take circumscribing 2^n -gons, rather than inscribed ones?



$$\begin{aligned} \text{Area (outer polygon)} &= 2^n \cdot \frac{R \cdot R \tan \frac{\theta_n}{2}}{2} = \frac{R}{2} \left(2 \cdot R \tan \frac{\theta_1}{2} + \dots + 2 \cdot R \tan \frac{\theta_n}{2} \right) \\ &= \frac{R}{2} \tilde{P}_n \quad \text{where } \tilde{P}_n = \text{outer perimeter} \end{aligned}$$

$$\text{We get } \text{Area (polygon)} \leq \text{Area (circle)} \leq \text{Area (outer polygon)} \quad (\star)$$

$$\frac{h_n}{2} \tilde{P}_n \leq \text{Area (circle)} \leq \frac{R}{2} \tilde{P}_n$$

$$\downarrow \pi R^2$$

$$\downarrow \frac{R}{2} 2\pi R = \pi R^2$$

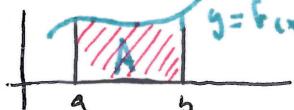
We conclude by "Sandwich Lemma" that $\text{Area (circle)} = \pi R^2$.

We can use this to approximate π up to any digit we want! Set $R=1$.

$$\left. \begin{aligned} h_n &= \cos \frac{\pi}{2^{n+1}} \\ b_n &= 2 \sin \frac{\pi}{2^{n+1}} \Rightarrow P_n = 2^{n+1} \sin \frac{\pi}{2^{n+1}} \\ \tilde{P}_n &= 2^n 2 \tan \frac{\pi}{2^{n+1}} \end{aligned} \right\} \begin{aligned} h_n \tilde{P}_n &= 2^n \cos \frac{\pi}{2^{n+1}} \sin \frac{\pi}{2^{n+1}} \\ &= 2^{n-1} \sin \left(\frac{2\pi}{2^{n+1}} \right) = 2^{n-1} \sin \left(\frac{\pi}{2^n} \right) \end{aligned}$$

We replace in (\star) to get $2^{n-1} \sin \left(\frac{\pi}{2^n} \right) \leq \pi \leq 2^n \tan \left(\frac{\pi}{2^{n+1}} \right)$ \square

- Knowing the Perimeter (Length) of the circle gives us the formula for the Area
- We'll use inner & outer approximations to compute areas under the graph of a positive function, replacing triangles with rectangles \Rightarrow Riemann sums..



$$A := \int_a^b f(x) dx$$

§2. Summation notation & some sums:

2 notations for sums : Σ (Greek S) \int (integral sign)

$$\text{Notation: } a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$$

$$\text{Example: } \sum_{k=1}^5 k = 1+2+3+4+5 = 15 \quad , \quad \sum_{i=1}^4 (-1)^i \frac{1}{i} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12}$$

$$\sum_{j=2}^7 j^2 = 4+9+16+25+36+49 = 139 = \frac{7 \cdot 8 \cdot 15}{6} - 1$$

We would like to have closed formulas whenever possible.

$$\text{Prop 1: } \sum_{k=1}^n k = \frac{n(n+1)}{2} \quad ; \quad \text{Prop 2: } \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{array}{c} \text{Proof 1} \\ (\text{Gauss}) \end{array} \quad \begin{array}{c} 1+2+\dots+n \\ + n+(n-1)+\dots+1 \\ \hline (n+1)+(n+1)+\dots+(n+1) \\ \text{n times} \end{array} \quad \text{so } 2 \sum_{k=1}^n k = n(n+1) \quad \square$$

Proof 2 (Which generalizes to higher powers see HW5)

$$(k+1)^2 = k^2 + 2k + 1 \quad \text{or} \quad (k+1)^2 - k^2 = 2k + 1.$$

We use telescoping / cancellation :

$$\begin{aligned} (2^2 - 1^2) + (3^2 - 2^2) + \dots + ((n+1)^2 - n^2) &= (n+1)^2 - 1^2 = n^2 + 2n \\ &= (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + \dots + (2n + 1) = \sum_{k=1}^n (2k + 1) = 2 \sum_{k=1}^n k + n \end{aligned}$$

$$\text{So } 2 \sum_{k=1}^n k = n^2 + n = n(n+1) \quad \square$$

$$\text{In general: } (k+1)^m - k^m = m k^{m-1} + \binom{m}{2} k^{m-2} + \dots + \binom{m}{m-1} k + 1$$

If we know how to add up $\sum_{k=1}^n k$, $\sum_{k=1}^n k^2$, ..., $\sum_{k=1}^n k^{m-2}$, and $\sum_{k=1}^n (k+1)^m - k^m = (n+1)^m - 1$, then we can give a formula for $\sum_{k=1}^n k^{m-1}$.

Example where $m=2$ used to find $\sum_{k=1}^n k^1$ and Prop 3 $\sum_{k=1}^n k^3 = \frac{(n(n+1))^2}{4}$.