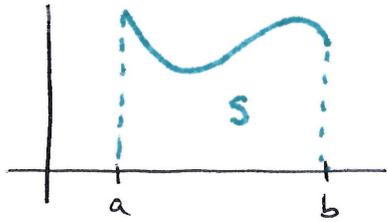


Lecture XXI: §6.4 The Area under a curve. Definite Riemann integrals  
 §6.5. The Computation of areas as limits

§1. Area under a curve:

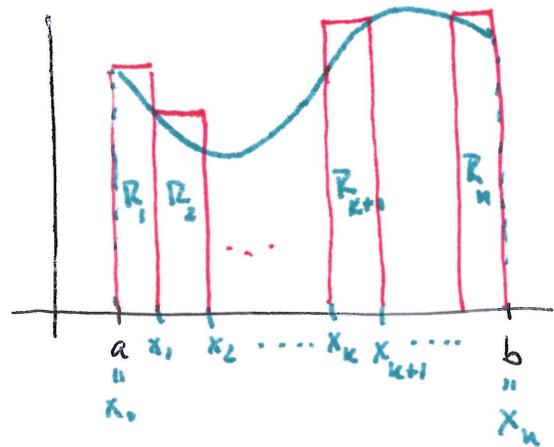
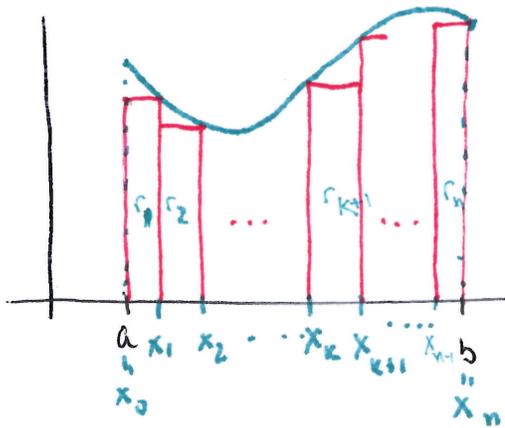
GOAL: Given a (cont.) function  $f: [a, b] \rightarrow \mathbb{R}_{\geq 0}$ , find the area of the region  $S$  that lies under the curve  $y = f(x)$  & above the  $x$ -axis.



We compute  $\text{Area}(S)$  by 2 approximations:

- (1) overestimate (Upper Riemann Sums) (circum)
- (2) underestimate (Lower Riemann Sums) (exhaust)

Both approximations will use rectangles with base in the  $x$ -axis of length  $\Delta x_k$  small & height either a min/max value of  $f$  restricted to the base:



STEP 1 Subdivide the segment  $[a, b]$  into  $n$  pieces of (the same) small length by means of  $n+1$  points  $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

Length of each piece  $:= \Delta x_k = x_{k+1} - x_k$  for  $k=0, \dots, n-1$

For simplicity: we assume all lengths are the same  $= \Delta x = \frac{b-a}{n}$

STEP 2: Build 2 rectangles for each segment  $[x_k, x_{k+1}]$ :

- "Lower rectangle"  $r_k$  of base  $[x_k, x_{k+1}]$  & height  $m_k := \min \{ f(x) : x_k \leq x \leq x_{k+1} \}$
- "Upper rectangle"  $R_k$  of base  $[x_k, x_{k+1}]$  & height  $M_k := \max \{ f(x) : x_k \leq x \leq x_{k+1} \}$

NOTE:  $m_k$  &  $M_k$  need to be in  $\mathbb{R}$ . This is the case, for example, if  $f$  is continuous.

$\text{Area}(r_k) = m_k \cdot \Delta x_k$  &  $\text{Area}(R_k) = M_k \Delta x_k$

STEP 3: The area covered by  $\{r_1, r_2, \dots, r_n\}$  &  $\{R_1, R_2, \dots, R_n\}$

satisfy:  $A(r) = \sum_{i=1}^n \text{Area}(r_i) \leq \text{Area}(S) \leq \sum_{i=1}^n \text{Area}(R_i) = A(R)$

Note: As we increase  $n$ , we decrease the length of  $\Delta x$ , so  $A(r)$  grows &  $A(R)$  gets smaller. Both have the same limit = Area(S).

*This is the theorem!*

Note 2: By the Extremal Value Thm, assuming  $f$  is continuous, we have

$m_k = f(\underline{x}_k)$  for since  $x_{k-1} \leq \underline{x}_k \leq x_k$

$M_k = f(\bar{x}_k)$  —————  $x_{k-1} \leq \bar{x}_k \leq x_k$

So  $A(r) = \sum_{k=1}^n f(\underline{x}_k) \Delta x_k$

$A(R) = \sum_{k=1}^n f(\bar{x}_k) \Delta x_k$

Furthermore:  $A(r) \leq \sum_{k=1}^n f(x_k^*) \Delta x_k \leq A(R)$

for any  $x_k^*$  in  $[x_{k-1}, x_k]$

Riemann Sum

& don't need all lengths  $\Delta x_k$  to be the same.

Saying intervals are shrinking to 0 is equivalent to  $\max \Delta x_k \rightarrow 0$  and if this happens, then  $n \rightarrow \infty$ .

We conclude  $\text{Area}(S) = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$

for any choice of points  $x_k^*$  in  $[x_{k-1}, x_k]$  for  $k=1, \dots, n$ .

Eg: pick  $x_k^* = x_{k-1}$  (left-most point)

$x_k^* = x_k$  (right-most point)

$x_k^*$  with  $f(x_k^*) = m_k$  (min value)

—————  $f(x_k^*) = M_k$  (max value)

Definition: We define the definite integral as the Area of  $S$ , i.e.

$\int_a^b f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \text{Area}(S) \text{ in } \mathbb{R}$

- $a$  = lower limit of integration
- $b$  = upper —————
- $f(x)$  = integrand,  $x$  = variable of integration.

We say  $f$  is integrable on  $[a, b]$  if the limit exists & is independent of all our choices  $\{x_1, \dots, x_n, x_1^*, \dots, x_n^*\}$

Note: If  $f$  is continuous,  $m_k$  &  $M_k$  are close when  $\Delta x_k$  is sufficiently small, so  $\int_a^b f(x) dx$  is well-defined by the Squeeze Lemma.

This is the theorem!

Q: What if  $f$  takes negative values? See § 6.7.

Remark: Not every function is integrable!

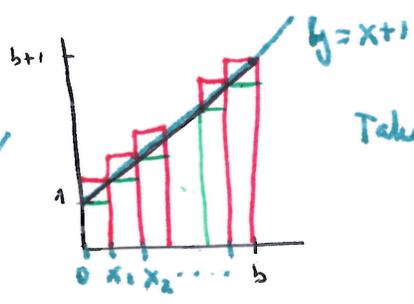
Eg:  $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$   $\implies$  Lower sums:  $m_k = 0$  for all  $k$   
Upper sums:  $M_k = 1$

so  $A(L) = 0$  &  $A(R) = 1$  they don't have the same limit

So  $f$  is not integrable (Note that  $f$  is not continuous!)

§ 8 Examples:

①  $f: [0, b] \rightarrow \mathbb{R}$   $f(x) = x+1$  cont ✓



Take  $\Delta x_k = \Delta x = \frac{b-0}{n}$

Points  $x_0 = 0, x_1 = \frac{b}{n}, x_2 = \frac{2b}{n}, \dots, x_n = b$

Lower rectangles:  $\bar{x}_k = x_{k-1} \implies M_k = f(\bar{x}_k) = x_{k-1} + 1$   
Upper rectangles:  $\bar{x}_k = x_k \implies M_k = f(\bar{x}_k) = x_k + 1$  } because  $f$  is increasing

$$\begin{aligned} \text{So } A(L) &= \sum_{k=1}^n m_k \Delta x = \sum_{k=1}^n \frac{b}{n} (x_{k-1} + 1) = \frac{b}{n} \sum_{k=1}^n (x_{k-1} + 1) \\ &= \frac{b}{n} \left( \frac{b}{n} \sum_{k=1}^n (k-1) + \sum_{k=1}^n 1 \right) = \frac{b}{n} \left( \frac{b}{n} \sum_{k=1}^{n-1} k + n \right) \\ &= \frac{b^2}{n^2} \frac{(n-1)n}{2} + b = b + \frac{b^2}{2} \left( \frac{n^2 - n}{n^2} \right) = b + \frac{b^2}{2} \left( 1 - \frac{1}{n} \right) \end{aligned}$$

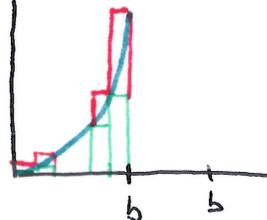
Similarly  $A(R) = \sum_{k=1}^n M_k \Delta x = \sum_{k=1}^n \frac{b}{n} (x_k + 1) = \frac{b}{n} \left( \frac{b}{n} \sum_{k=1}^n k + \sum_{k=1}^n 1 \right)$

$$= \frac{b^2}{n^2} \frac{n(n+1)}{2} + b = \frac{b^2}{2} \left( 1 + \frac{1}{n} \right)$$

Note  $\lim_{n \rightarrow \infty} A(L) = \lim_{n \rightarrow \infty} \left( b + \frac{b^2}{2} \left( 1 - \frac{1}{n} \right) \right) = b + \frac{b^2}{2}$   
 $\lim_{n \rightarrow \infty} A(R) = \lim_{n \rightarrow \infty} \left( b + \frac{b^2}{2} \left( 1 + \frac{1}{n} \right) \right) = b + \frac{b^2}{2}$  } same limit  
 $= \int_0^b (1+x) dx$

②  $f: [0, b] \rightarrow \mathbb{R}^2$   $f(x) = x^2$  cont ✓

$\Delta x_k = \frac{b-0}{n}$



Points:  $x_0 = 0$   
 $x_1 = \frac{b}{n}, x_2 = \frac{2b}{n}, \dots, x_k = \frac{kb}{n}$

$f$  increasing so  $\underline{x}_k = x_{k-1}$  &  $\overline{x}_k = x_k$ .

Lower rectangles:  $A(L) = \sum_{k=1}^n m_k \Delta x = \sum_{k=1}^n (x_{k-1})^2 \frac{b}{n} = \sum_{k=1}^n \frac{(k-1)^2 b^2}{n^2} \frac{b}{n}$   
 $= \frac{b^3}{n^3} \sum_{k=1}^n (k-1)^2 = \frac{b^3}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{b^3}{n^3} \frac{(n-1)n(2n-1)}{6}$

Upper rectangles:  $A(R) = \sum_{k=1}^n M_k \Delta x = \sum_{k=1}^n x_k^2 \frac{b}{n} = \sum_{k=1}^n \frac{k^2 b^2}{n^2} \frac{b}{n}$   
 $= \frac{b^3}{n^3} \sum_{k=1}^n k^2 = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$

$\lim_{n \rightarrow \infty} A(L) = \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) = \frac{b^3}{3}$

$\lim_{n \rightarrow \infty} A(R) = \lim_{n \rightarrow \infty} \frac{b^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = \frac{b^3}{3}$  } same limit!

So  $\int_0^b x^2 dx = \frac{b^3}{3}$ .