

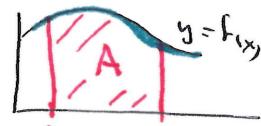
## Lecture XXII

§6.6 The Fundamental Theorem of Calculus

§6.7 Properties of definite integrals.

Recall  $A = \int_a^b f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$  (if  $f$  is continuous, this limit exists!)

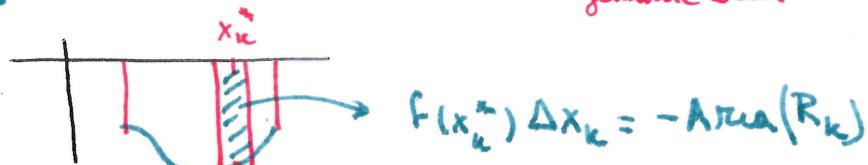
Example (for last time)  $\int_0^b x dx = \frac{b^2}{2}$ ,  $\int_0^b x^2 dx = \frac{b^3}{3}$



geometric area

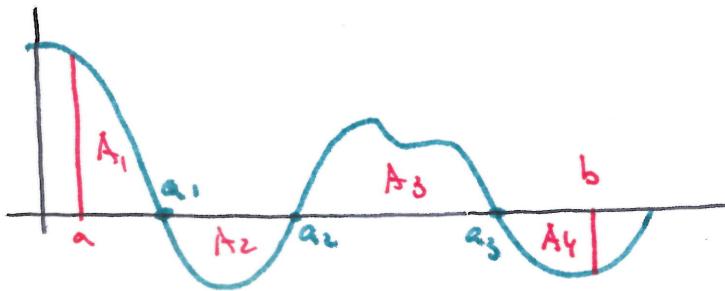
### §1 Algebraic vs geometric area

Q: What if  $f(x) \leq 0$ ?



A If  $f(x) \leq 0$ , then  $\int_a^b f(x) dx = -\text{Area} = - \int_a^b (-f(x)) dx$ .

In general: If  $f(x)$  goes above & below x-axis



• "Signed area":  $\int_a^b f(x) dx = A_1 - A_2 + A_3 - A_4$   
Algebraic area  
• Area =  $A_1 + A_2 + A_3 + A_4 = \int_a^b |f(x)| dx$   
 $\geq 0$  & cont.

Q: How do we compute this? Find the zeros of  $f = a_1, a_2, a_3$

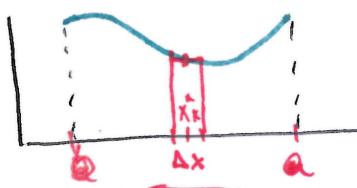
$$A_1 = \int_a^{a_1} f(x) dx, \quad A_2 = \int_{a_1}^{a_2} -f(x) dx, \quad A_3 = \int_{a_2}^{a_3} f(x) dx, \quad A_4 = \int_{a_3}^b f(x) dx$$

In general:  $a = a_0 < a_1 < \dots < a_{n-1} < a_n = b$   
zeros of  $f$ .

$$A_i = \int_{a_{i-1}}^{a_i} |f(x)| dx \quad \text{for } i=1, \dots, n \text{ as } \text{Signed area} = \sum_{i=1}^n \underbrace{\text{sign}_f}_{[a_{i-1}, a_i]} \cdot A_i.$$

### §2 Miscellaneous properties

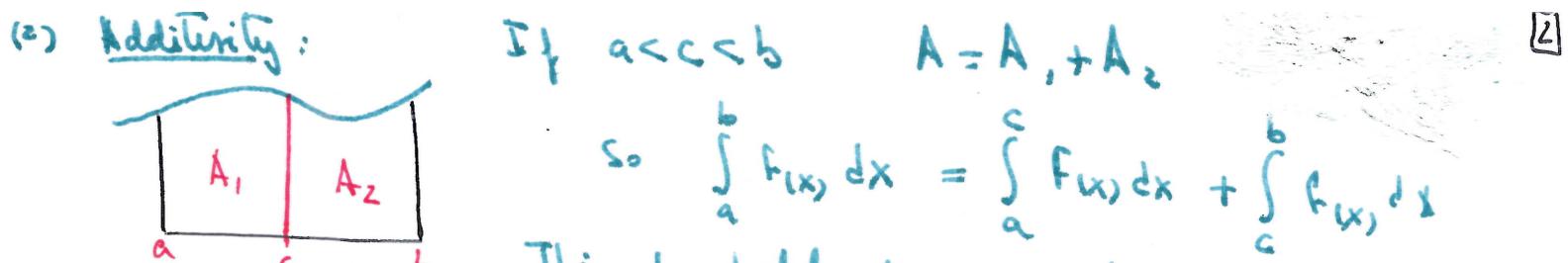
(d) If  $b < a$ , what does  $\int_a^b f(x) dx$  mean?



goes from a to b, so  $\Delta x_k = x_{k+1} - x_k < 0$ .

Then  $\int_a^b f(x) dx = - \int_b^a f(x) dx$  (Intuition:  $\int_a^a f(x) dx = 0$ )

$\begin{cases} + & \text{if } f \geq 0 \\ - & \text{if } f \leq 0 \end{cases}$



- This also holds for any order between  $a, b, c$ .
- Also true for algebraic / signed area (by Property (1))

(3) Scalar Mult.  
f cont.,  $\lambda$  in  $\mathbb{R}$

$$\int_a^b \lambda f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \lambda f(x_k^*) \Delta x_k$$

because limit exists &  $\lambda$  is fixed

$$= \lambda \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \lambda \int_a^b f(x) dx.$$

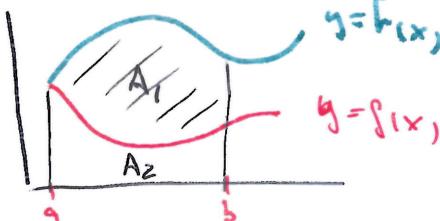
Addition: Similarly

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

[“Limit of a sum is the sum of the limits when they both exist”].

(4) Areas between curves: Replace the  $x$ -axis by any curve  $y = g(x)$ .

(I)



$$f(x) \geq g(x) \text{ in } [a, b]$$

$$A = A_1 + A_2$$

$$\Rightarrow A_1 = A - A_2 = \int_a^b f(x) dx - \int_a^b g(x) dx$$

$$A_1 = \int_a^b (f(x) - g(x)) dx$$

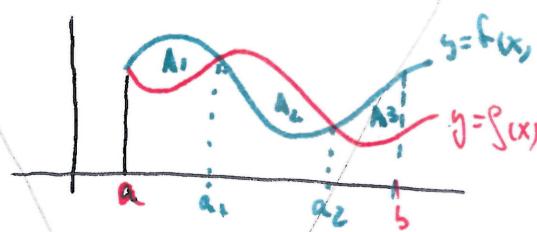
Note:  $g(x) \leq f(x)$  given  $\int_a^b g(x) dx \leq \int_a^b f(x) dx$

Why?

$$\int_a^b g(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n g(x_k^*) \Delta x_k$$

$$\int_a^b f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

(II)



$$\text{Given Area} = A_1 + A_2 + A_3$$

$$= \int_a^b |f(x) - g(x)| dx$$

$$\text{Signed Area} = A_1 - A_2 + A_3$$

$$= \int_a^b f(x) - g(x) dx$$

Q: How to find  $A_1, A_2, A_3$ ?

Fnd. points  $a_1, a_2$ , where.

$$f(x) = g(x)$$

$$A_i = \int_{a_i}^{a_{i+1}} |f(x) - g(x)| dx$$

$$(A_1 = \int_a^{a_1} f(x) - g(x) dx, A_2 = \int_{a_1}^{a_2} g(x) - f(x) dx)$$

$$A_3 = \int_{a_2}^b f(x) - g(x) dx$$

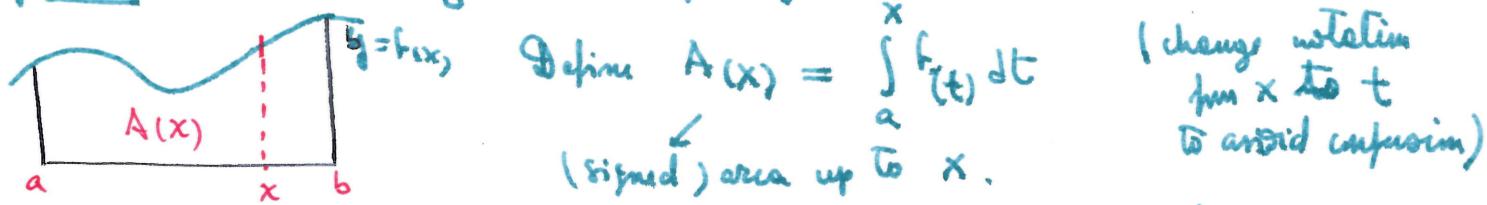
### §3 Fundamental Theorem of Calculus [Relates Differential & Integral Calculus]

Fund Thm: Assume  $f: [a, b] \rightarrow \mathbb{R}$  is continuous, and let  $F(x)$  be any antiderivative of  $f$ ,  $F(x) = \int f(x) dx$  & the x-axis. Then, the area under the curve  $y = f(x) \geq 0$  from  $a$  to  $b$  is

$$\int_a^b f(x) dx = F(b) - F(a) =: F \left|_a^b \quad (\text{or } F(x)]_a^b\right)$$

- Eg:  $\int_0^b x^2 dx = \frac{x^3}{3} \Big|_0^b = \frac{b^3}{3} - 0 = \frac{b^3}{3}$ .

- Proof idea (Newton-Leibniz) For simplicity, we assume  $f(x) \geq 0$

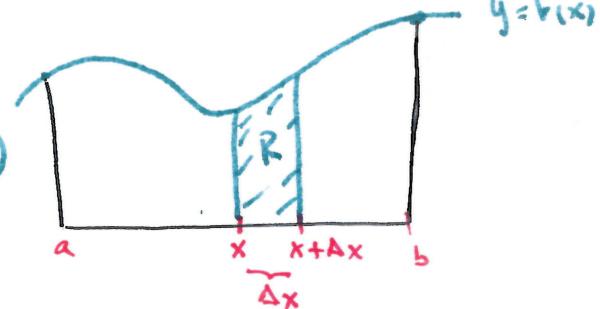


It seems that  $A(x)$  is a smooth function whenever  $f$  is continuous. We want to compute  $A'(x)$ . Use method of increments:

$$\frac{dA}{dx} = \lim_{\Delta x \rightarrow 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}$$

What is  $A(x + \Delta x) - A(x)$ ? =  $\text{Area}(R)$

= Area under the curve  $y = f(x)$  between  $x \approx x + \Delta x$



We want to estimate  $\text{Area}(R)$ : Pick  $m = \min_{[x, x+\Delta x]} \{f(u)\}$ ,  $M = \max_{[x, x+\Delta x]} \{f(u)\}$

$$m \Delta x \leq \text{Area}(R) \leq M \Delta x$$

$$f(\underline{u}) \leq \frac{\text{Area}(R)}{\Delta x} \leq f(\bar{u})$$

but, so there are  $\{m = f(\underline{u})\}$

by the EVT.

Intermediate Value Theorem,

Since  $f$  is continuous in  $[x, x + \Delta x]$  by the

$$\frac{\text{Area}(R)}{\Delta x} = f(x^*) \text{ for some } x^* \text{ in } [x, x + \Delta x]$$

So  $\frac{\text{Area}(R)}{\Delta x} = f(x^*) \xrightarrow[\Delta x \rightarrow 0]{} f(x)$  because  $x^* \xrightarrow[\Delta x \rightarrow 0]{} x$  &  $f$  is cont.

In conclusion  $\frac{dA}{dx} = f(x)$  &  $A$  is indeed differentiable!

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- $A_{(x)}$  is an antiderivative of  $f(x)$  &  $A_{(x)} = \int_a^x f(t) dt$
  - By "uniqueness" we know  $A_{(x)} = F(x) + C$  for some constant  $C$ . We find it by evaluating at  $x=a$ .

$$0 = A(a) = F(a) + C \quad \text{so } C = -F(a).$$

$$A(x) = \int_a^x f(t) dt = F(x) - F(a) \quad \text{for all } x.$$

$$\text{Evaluate at } x=b \quad \& \text{ we get } A(b) = \int_a^b f(t) dt = F(b) - F(a).$$

Remark: The result is true for ANY antiderivative  $F$ , since if we change it to  $G(x) = F(x) + C'$ , then  $\bar{F}(b) - G(a) = F(b) + C' - (F(a) + C')$

Note: The proof works for any antif, replacing "area" by "signed area"  $= \bar{F}(b) - \bar{F}(a)$ .

Corollary: If  $F$  is ant  $\left( \int_a^x f(t) dt \right)' = f(x)$ .

$$\cdot \left( \int_x^a f(t) dt \right)' = -f(x) \quad (\underline{\text{Exercise: Why?}})$$

• If  $f$  is ant,  $u(x)$  is differentiable :  $\frac{d}{dx} \int_a^{u(x)} f(t) dt = \frac{d}{dx} A_{(u(x))}$

$$= \frac{dA}{du} \cdot \frac{du}{dx} = f(u) \cdot u'(x).$$

Chain rule