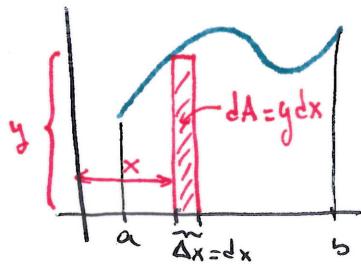


Lecture XXIII: §7.1 The intuitive meaning of integration
 §7.2 The area between two curves

Recall:



$$A(R) = \int_a^b y \, dx = \boxed{\int_a^b f(x) \, dx = dA} \quad \text{(differential element of area)}$$

$$A(x) = \int_a^x f(t) \, dt \quad \& \quad A'(x) = f(x) \quad (\text{proof of FTC})$$

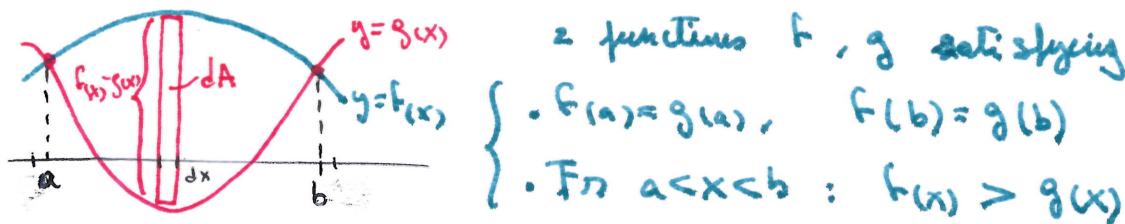
Area = add areas of vertical strips.

§1 The area between two curves:

GOAL: Compute the area of a region bounded by 2 smooth curves,

[So far: one of the curves was the x-axis. Now we want other curves!]

Easiest Examples



• Height of each strip = $f(x) - g(x)$

• Length of base = dx

• Element of Area = $(f(x) - g(x)) dx =: dA$

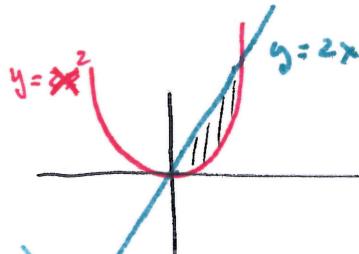
$$\text{Area} = \int_a^b (f(x) - g(x)) \, dx.$$

Example 1: $f(x) = x^2$, $g(x) = 2x$.

Step 1: Draw the curves & find a, b

$$f(x) = x^2 = 2x = g(x) \text{ gives } x^2 - 2x = x(x-2) = 0 \quad \text{so } x=0 \text{ & } x=2$$

$$\boxed{a=0, \quad b=2}$$



Step 2: Check that between a & b $f(x) \geq g(x)$

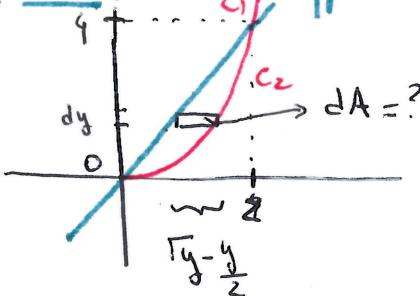
$$\text{Check: } x^2 \leq 2x \quad \Rightarrow \quad 0 \leq x \leq 2$$

Since $x > 0$ divide by x : get $x \leq 2$ ✓

Step 3: Use Fund Theorem of Calculus:

$$A = \int_a^b dA = \int_a^b (f(x) - g(x)) \, dx = \int_0^2 (2x - x^2) \, dx = \left. x^2 - \frac{x^3}{3} \right|_{x=0}^{x=2} = \frac{4 - \frac{8}{3}}{3} = \boxed{\frac{4}{3}}$$

Question: What happens if we use horizontal strips?



- Write 2 curves as functions of y

- $C_1: y = 2x \Rightarrow x = \frac{y}{2}$

- $C_2: y = x^2 \Rightarrow x = \sqrt{y}$
 $y > 0$

- Bounds: $y=0$ & $y=4$

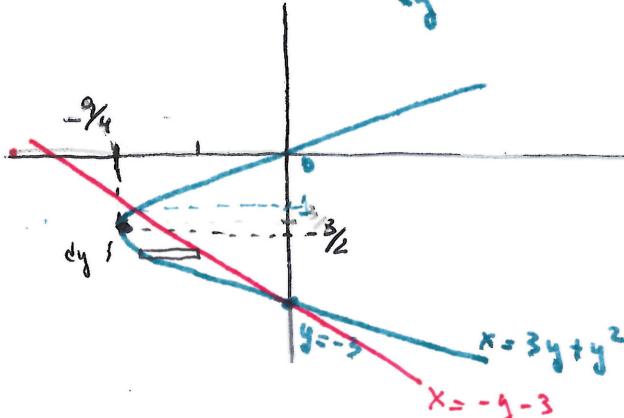
$$dA = \left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$A = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right) dy = \frac{2}{3}y^{3/2} - \frac{y^2}{4} \Big|_{y=0}^{y=4} \\ = \frac{2}{3} \cdot 4^{3/2} - \frac{16}{4} = \frac{2 \cdot 8}{3} - \frac{16}{4} = \frac{16}{12} = \boxed{\frac{4}{3}}$$

Example 2: Find the area between the curves $x = 3y + y^2$ (parabola) & $x + y + 3 = 0$ (line)

- Seems easier to use horizontal strips.

Vertex of parabola $\frac{dx}{dy} = 3 + 2y = 0 \Rightarrow y = -\frac{3}{2} \Leftarrow x = 3 \cdot \left(-\frac{3}{2}\right) + \frac{9}{4} = -\frac{9}{4}$



y-intercept: $x=0 = 3y+y^2 = y(3+y)$
 $\therefore y=0 \text{ or } y=-3$

Line: x-intercept $x+0+3=0 \Rightarrow x=-3 \quad y=0$
y-intercept $0+y+3=0 \Rightarrow y=-3 \quad x=0$

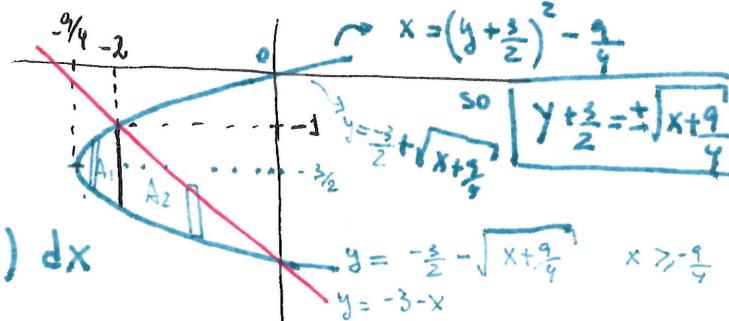
Check vertex: $x = -\frac{3}{2} + 3 = 0 \quad \text{the vertex is to the left of the line}$
 $x = -\frac{3}{2} > -\frac{9}{4} \Rightarrow$ so to the left of the line

- Check intersections: $x = 3y + y^2 = -3 - y$ gives $y^2 + 4y + 3 = 0$

Solutions: $y = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm 2}{2} = -2 \pm 1 \quad \begin{matrix} -1 \\ -3 \end{matrix}$

- $A = \int_{-3}^{-1} ((-y-3) - (3y+y^2)) dy = \int_{-3}^{-1} -y^2 - 4y - 3 dy = \left[-\frac{y^3}{3} - 2y^2 - 3y \right]_{y=-3}^{y=-1}$

$$= \frac{1}{3} - 2 + 3 - (9 - 18 + 9) = \boxed{\frac{4}{3}}$$



Q: What if we do vertical strips?

$$A = A_1 + A_2$$

$$A_2 = \int_{-2}^0 \left((-3-x) - \left(-\frac{3}{2} - \sqrt{x+\frac{9}{4}}\right)\right) dx$$

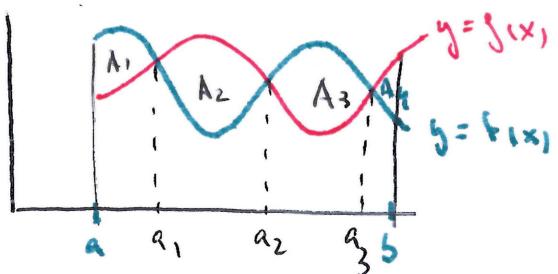
$$A_2 = \int_{-2}^{-\frac{1}{2}} \left(-\frac{3}{2}x - x + \sqrt{x + \frac{9}{4}} \right) dx = \left. -\frac{3}{2}x^2 - \frac{x^2}{2} + \frac{2}{3} \left(x + \frac{9}{4} \right)^{3/2} \right|_{x=-2}^{x=-\frac{1}{2}} = \frac{2}{3} \left(\frac{3}{2} \right)^3 - \left(+1 + \frac{2}{3} \cdot \frac{1}{8} \right) = \frac{9}{4} - \frac{13}{48} = \frac{14}{48}$$

$$A_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{3}{2}x + \sqrt{x + \frac{9}{4}} \right) - \left(-\frac{3}{2}x - \sqrt{x + \frac{9}{4}} \right) dx = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{x + \frac{9}{4}} dx$$

$$= \frac{4}{3} \left(x + \frac{9}{4} \right)^{3/2} \Big|_{x=-\frac{1}{2}}^{x=\frac{1}{2}} = \frac{4}{3} \left(\frac{1}{4} \right)^{3/2} = \frac{4}{3} \cdot \frac{1}{8} = \frac{4}{24} = \frac{1}{6}$$

TOTAL: $\frac{1}{6} + \frac{1}{6} = \frac{8}{6} = \frac{4}{3}$ (same as earlier!)

Multiple crossings:



- Geometric Area = $A_1 + A_2 + A_3 + A_4 = \int_a^b |f(x) - g(x)| dx$
- Signed Area = $A_1 - A_2 + A_3 - A_4 = \int_a^b f(x) - g(x) dx.$

Q: How do we find A_1, A_2, A_3 & A_4 ?

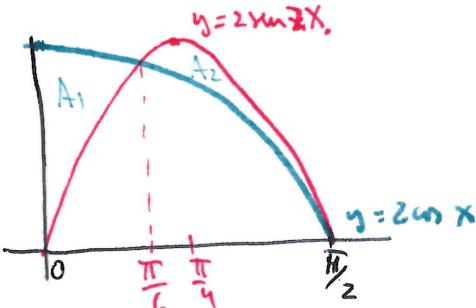
• Find all crossings of the curves between $x=a$ & $x=b$ ($f(x) = g(x)$ for $a \leq x \leq b$)

- $A_1 = \int_a^{a_1} \underbrace{f(x) - g(x)}_{\geq 0} dx$
- $A_3 = \int_{a_2}^{a_3} f(x) - g(x) dx$
- $A_2 = \int_{a_1}^{a_2} \underbrace{g(x) - f(x)}_{\geq 0} dx = \int_{a_1}^{a_2} |f(x) - g(x)| dx$
- $A_4 = \int_{a_3}^b g(x) - f(x) dx.$

Example 3 Find the area between $y = 2\cos x$ & $y = 2\sin 2x$ in $[0, \frac{\pi}{2}]$

STEP 1: Draw the curves & find the crossings.

$$A = A_1 + A_2$$



$$2\cos x = 2\sin 2x$$

$$\cos x = \sin 2x$$

$$\cos x = 2\cos x \sin x$$

$$\text{so } \cos x = 0 \quad \text{or} \quad 1 = 2\sin x$$

$$\boxed{x=0} \quad ; \quad \boxed{\frac{1}{2} = \sin x}$$

$$\text{so } \boxed{x = \frac{\pi}{6}}$$

To check which curve lies above, evaluate at any pt between the crossings

$$A_1 = \int_0^{\frac{\pi}{2}} (2\cos x - 2\sin 2x) dx = 2\sin x + \cos 2x \Big|_0^{\frac{\pi}{2}} = 2\left(\frac{1}{2} + \frac{1}{2}\right) - (0+1) = \frac{1}{2},$$

$$A_2 = \int_{\frac{\pi}{2}}^{\pi} -(2\cos x - 2\sin 2x) dx = - (2\sin x + \cos 2x) \Big|_{\frac{\pi}{2}}^{\pi} = - (2 + (-1)) + \left(2\left(\frac{1}{2} + \frac{1}{2}\right)\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

$$\text{TOTAL : } A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = \boxed{1}.$$

$$\text{Signed area} = A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = \boxed{0}.$$