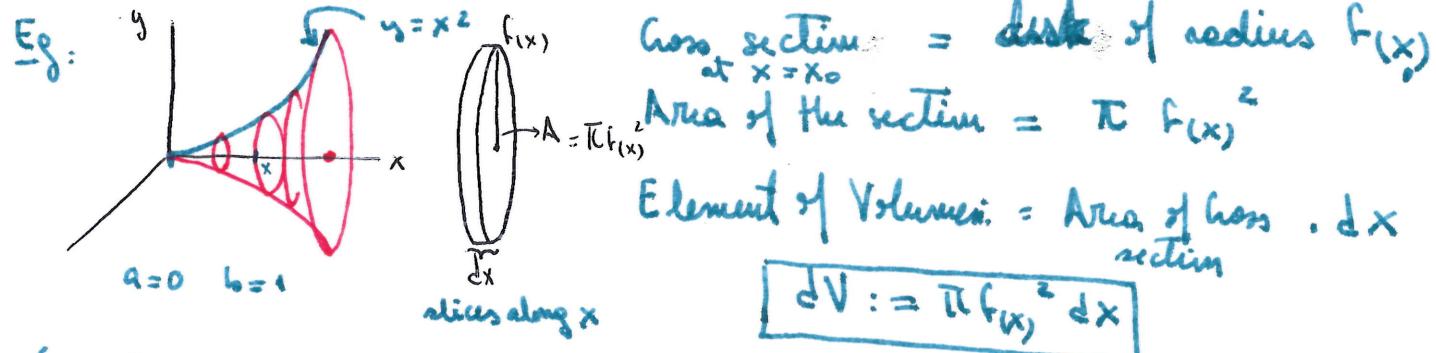


Lecture XXIV §7.3 Volumes: The Disk Method.

Solid of revolution:

- Idea: - Give $y = f(x)$ with $a \leq x \leq b$ & assume f is positive (e.g. $f_{xx} = x^2$)
 • Rotate the graph about the x -axis. We get a solid of revolution R
 • Question: What's the Volume of R ?



Cover R with these vertical slices a lit $dx \rightarrow 0$:

$$Vol(R) = \int_a^b dV = \int_a^b \pi f(x)^2 dx$$

$$Eg: Vol = \int_0^1 \pi x^2 dx = \pi \frac{x^3}{3} \Big|_0^1 = \frac{\pi}{3}$$

$$\begin{aligned} \text{Why? } Vol(R_x)' &= \lim_{\Delta x \rightarrow 0} \frac{Vol(R_{x+\Delta x}) - Vol(R_x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\int_a^{x+\Delta x} \pi f(t)^2 dt - \int_a^x \pi f(t)^2 dt}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} \pi f(t)^2 dt}{\Delta x} \stackrel{?}{=} \pi f(x)^2. \end{aligned}$$

$$\text{As with areas: } \pi f(x)^2 \leq \pi f(t)^2 \leq \pi f(\bar{x})^2$$

$$\pi f(x)^2 \Delta x \leq \int_x^{x+\Delta x} \pi f(t)^2 dt \leq \pi f(\bar{x})^2 \Delta x$$

$$\begin{aligned} \pi f(x)^2 &\leq \frac{\int_x^{x+\Delta x} \pi f(t)^2 dt}{\Delta x} \leq \pi f(\bar{x})^2 \\ \downarrow \Delta x \rightarrow 0 & \downarrow \Delta x \rightarrow 0 \\ \pi f(x)^2 & \end{aligned}$$

$$\begin{aligned} f(\bar{x}) &= \max \{ f(t) \} \\ &\text{for } t \in [x, x+\Delta x] \\ f(x) &= \min \{ f(t) \} \\ &\text{for } t \in [x, x+\Delta x] \end{aligned}$$

so limit of middle function exists & equals $\pi (f(x))^2$.

§2 Examples

① Sphere of radius a = rotate a half-circle

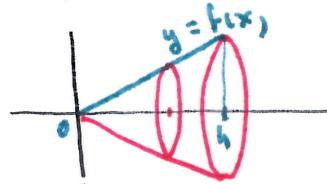
$$dV = \pi y^2 dx = \pi (a^2 - x^2) dx$$

$$\text{Vol} = \int_{-a}^a \pi (a^2 - x^2) dx = \left[\pi \left(a^2 x - \frac{x^3}{3} \right) \right]_{-a}^a = \pi \left(a^2 a - \frac{a^3}{3} - (-a^2 + \frac{a^3}{3}) \right) = \pi \left(2a^3 - \frac{2a^3}{3} \right) = \boxed{\frac{4}{3} \pi a^3}$$

② Cone of height h a radius of base $= r$.

Rotate a line segment

$$\begin{cases} y(0) = 0 \\ y(h) = r \end{cases} \Rightarrow y = \frac{r}{h} x = f(x)$$



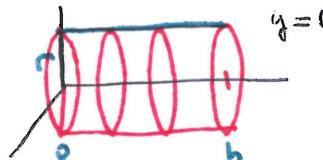
$$dV = \pi f(x)^2 dx = \pi \frac{r^2}{h^2} x^2 dx$$

$$\text{Vol} = \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \frac{x^3}{3} \Big|_0^h = \pi \frac{r^2 h^3}{h^2 \cdot 3} = \boxed{\frac{1}{3} \pi r^2 h^2}$$

③ Cylinder of radius r a height h

$$f(x) = r \text{ constant}$$

$$dV = \pi r^2 dx \quad \text{Vol} = \int_0^h \pi r^2 dx = \pi r^2 h$$

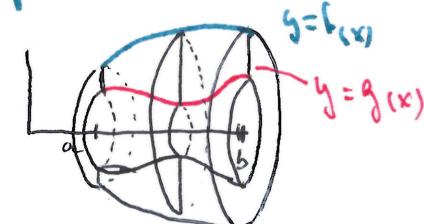


Note: $V_{\text{cone}} = \frac{1}{3} V_{\text{cyl}}$, $V_{\text{sphere}} = \frac{4}{3} V_{\text{cyl}}$
($h=2r$)

§3 Washer method:

Idea: Take 2 continuous functions with $g(x) \leq f(x)$ for $a \leq x \leq b$

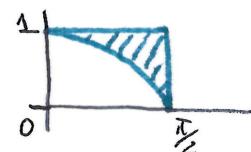
- Rotate both graphs.
- Cross section = washer
- = difference of 2 disks of radii $g(x) < f(x)$.



$$dV = \pi (f(x)^2 - g(x)^2) dx$$

$$\text{Vol} = \int_a^b dV = \int_a^b \pi (f(x)^2 - g(x)^2) dx$$

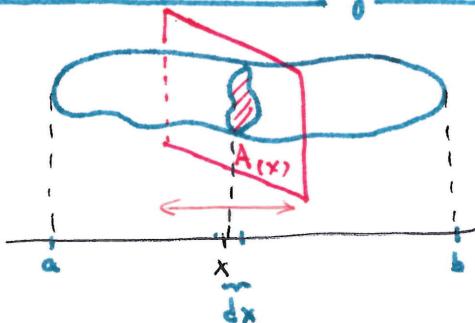
Example: $f(x) = 1$ $\delta(x) = \sqrt{\cos(x)}$ $0 \leq x \leq \frac{\pi}{2}$



$$dV = \pi (1^2 - \cos x) dx$$

$$\text{Vol} = \int_0^{\frac{\pi}{2}} \pi (1 - \cos x) dx = \pi (x - \sin x) \Big|_0^{\frac{\pi}{2}} = \pi \left(\frac{\pi}{2} - 1 \right) - (0 - 0) = \boxed{\frac{\pi(\pi - 2)}{2}}$$

§4 Volume via moving slices:

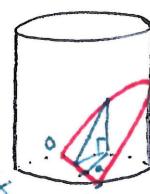


x-Slices: Area = $A(x)$

$$dV = A(x) dx \quad (\text{same idea: } \frac{V(x+\Delta x) - V(x)}{\Delta x} \rightarrow A)$$

$$\text{Vol} = \int_a^b dV = \int_a^b A(x) dx.$$

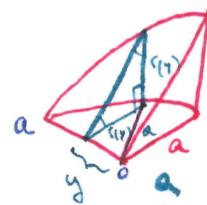
Example



45° plane cuts a cylinder through the center of its base.
of radius a

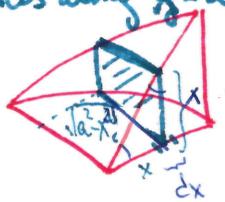
$$\text{Slices} = \Delta h(y) = r(y)$$

$$\begin{aligned} \text{Vol} &= 2 \text{Vol} \left(\text{one slice} \right) = 2 \int_0^a \text{Area} \left(\Delta h(y) \right) dy = 2 \int_0^a \sqrt{a^2 - y^2} \sqrt{a^2 - y^2} dy \\ &= \int_0^a (a^2 - y^2) dy = a^2 y - \frac{y^3}{3} \Big|_0^a = a^3 - \frac{a^3}{3} = \boxed{\frac{2}{3} a^3} \end{aligned}$$



$$r(y) = \sqrt{a^2 - y^2} \quad 0 \leq y \leq a$$

Slices along x-axis:



$$dV = x \sqrt{a^2 - x^2} dx$$

$$\begin{aligned} \text{Vol} &= 2 \int_0^a x \sqrt{a^2 - x^2} dx = -\int_{a^2}^0 \sqrt{u} du = -\frac{2}{3} u^{\frac{3}{2}} \Big|_{a^2}^0 \\ &\quad u = a^2 - x^2 \\ &\quad du = -2x dx \\ &= -\frac{2}{3} (0 - (a^2)^{\frac{3}{2}}) \\ &= \boxed{\frac{2}{3} a^3} \end{aligned}$$