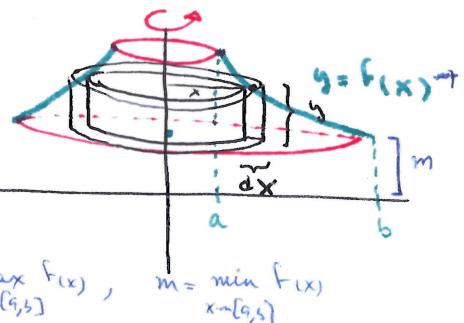


Lecture XXV § 7.4 Volumes: The method of cylindrical shells

§1 Cylinder shells:

Theorem: If $y = f(x) \geq 0$ for f continuous & decreasing on $a \leq x \leq b$ with $a \geq 0$.

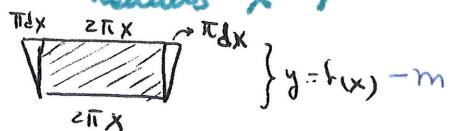
- Rotate about the y -axis to get a solid of revolution R
- Question: What's the volume of R ?



- If we can rewrite it as $x = g(y)$ $f(a) \leq y \leq f(b)$
Then we can use disk / washer method. (horiz strips)
- What if we don't have an explicit formula
to rewrite $y=f(x)$ as $x=g(y)$? no cylindrical shells!

Cylindrical Shell = Volume cylinder height $y = f(x) - m$ - Volume cylinder height & radius $x + dx$ & radius x .

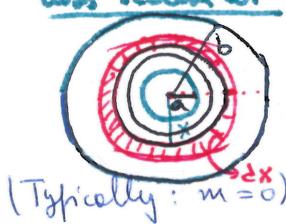
$$\approx dx \left(\text{length circle} \right) \text{height} = 2\pi x (f(x) - m) dx$$



$$dV = 2\pi x (f(x) - m) dx$$

$$Vol = \int_a^b dV = \int_a^b 2\pi x (f(x) - m) dx$$

Cross section at $y=m$ circle



area with concentric circles with radii increased by dx from a to b

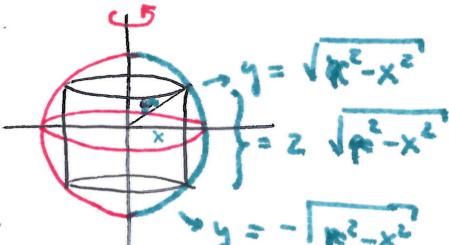
§2 Examples:

① Sphere of radius r

$2 Vol$ Half-sphere

$$a=0 \\ b=r$$

$$Vol = \int_0^r 4\pi x \sqrt{r^2 - x^2} dx = - \int_0^{r^2} 2\pi \sqrt{u} du = -\frac{4}{3}\pi u^{3/2} \Big|_0^{r^2} = -(0 - \frac{4}{3}\pi r^3) \\ dx = -2x dx$$



$$dV = 2\pi x \cdot 2\sqrt{r^2 - x^2} dx$$

$$0 \leq x \leq r$$

$$= -\int_0^{r^2} 2\pi \sqrt{u} du = -\frac{4}{3}\pi u^{3/2} \Big|_0^{r^2} = -(0 - \frac{4}{3}\pi r^3) \\ = \frac{4}{3}\pi r^3$$

② Sphere of radius r with a cylinder of radius $\frac{r}{2}$ cut through.

Same dV , but now $a = \frac{r}{2}$ $b = r$

$$Vol = \int_{\frac{r}{2}}^r 4\pi x \sqrt{r^2 - x^2} dx = - \int_{\frac{r}{2}}^r 2\pi \sqrt{u} du = -\frac{4}{3}\pi u^{3/2} \Big|_{\frac{r}{2}}^r = \frac{4}{3}\pi \frac{\sqrt{3}}{4} r^3 \\ = \frac{\sqrt{3}}{2}\pi r^3$$

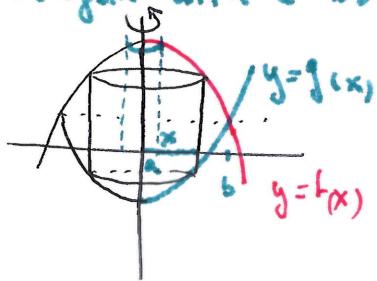


② Cone of height h & radius of base $= r$

$y = f(x)$ Line: $y = \frac{h}{r}(x-r) = h - \frac{h}{r}x = f(x)$ $0 \leq x \leq r$

 $dV = 2\pi x \left(h - \frac{h}{r}x\right) dx$
 $Vol = \int_0^r 2\pi x \left(h - \frac{h}{r}x\right) dx = \pi h x^2 - \frac{h\pi}{r} \frac{x^3}{3} \Big|_0^r = \pi h r^2 - \frac{\pi h r^2}{3} = \boxed{\frac{2}{3}\pi h r^2}$

③ Region with z boundary curves $y=f(x)$, $y=g(x)$



Height: $f(x) - g(x)$

f, g continuous

$f(x) \geq g(x)$

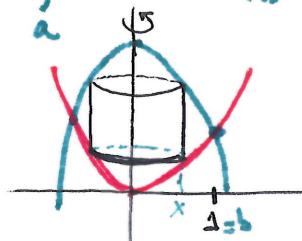
$dV = 2\pi x (f(x) - g(x)) dx$

$Vol = \int_a^b 2\pi x (f(x) - g(x)) dx$

Example: $\begin{cases} y = x^2 = g(u) \\ y = 2-x^2 = f(x) \end{cases}$

$dV = 2\pi x (2-x^2 - (x^2)) dx$

$= 2\pi x (2-2x^2) dx = 4\pi x (1-x^2) dx$



$a=0$

$g(b) = f(b)$

$b^2 = 2-b^2$

$2b^2 = 2$

$b^2 = 1 \quad b = \pm 1$

$\boxed{b=1}$

$Vol = \int_0^1 dV = \int_0^1 4\pi x (1-x^2) dx = -2 \int_1^0 \pi u du = -\pi u^2 \Big|_1^0 = \boxed{\pi}$

$u = 1-x^2$
 $du = -2x dx$

Alternative: disk method around y -axis

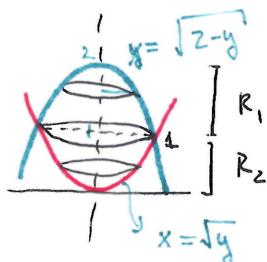
$Vol = Vol(R_1) + Vol(R_2)$

$Vol(R_1) \text{ via disks: } dV_1 = \pi x^2 dy = \pi (z-y) dy \quad 1 \leq y \leq z$

$Vol_1 = \int_1^z \pi (z-y) dy = \pi \left(zy - \frac{y^2}{2}\right) \Big|_1^z = \pi \left((z-z) - (z-\frac{1}{2})\right) = \frac{\pi}{2}$

$Vol(R_2): dV_2 = \pi x^2 dy = \pi y dy \quad 0 \leq y \leq 1$

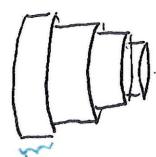
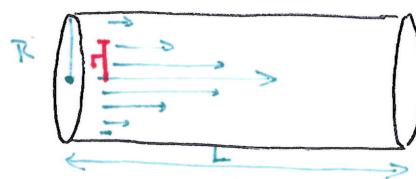
$Vol_2 = \int_0^1 \pi y dy = \frac{\pi y^2}{2} \Big|_0^1 = \frac{\pi}{2}$



$TOTAL = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$

§3 Blood flow

GOAL: Study blood flow along an artery.



$$\text{Laminar Flow} = \frac{(\text{varying})}{\text{speed}} + \text{area of cross section} \\ = \text{disk.}$$

(1) If constant speed s_0 : $F = s_0 \frac{\Delta P}{\eta L} = s_0 L \pi R^2$ volume of blood per unit time
 $\Delta F = s_0 (2\pi R)L$

(2) Known: speed is faster along center, gets close to 0 near the walls of the artery
 (by viscosity)

Speed depends on the distance to the center

$$v(r) = \frac{P}{4\eta L} (R^2 - r^2) \rightarrow \text{max. value} = \frac{PR^2}{4\eta L} \text{ at center}$$

where P = pressure difference between the ends of the artery

η = viscosity of blood

L = length

$$\begin{aligned} \text{Flow} &= \int_0^R s(r) \text{Area}(r) dr = \int_0^R \frac{P}{4\eta L} (R^2 - r^2) 2\pi r dr \\ &= \int_0^R \frac{P\pi}{2\eta L} (R^2 r - r^3) dr = \left. \frac{P\pi}{2\eta L} \left(\frac{R^2 r^2}{2} - \frac{r^4}{4} \right) \right|_0^R = \frac{P\pi}{2\eta L} \left(\frac{R^4}{2} - \frac{R^4}{4} \right) \end{aligned}$$

$$\boxed{\text{Flow} = \frac{\pi P R^4}{8\eta L}}$$

Poisuille's Law

$$\text{Exp values: } \left. \begin{array}{l} R = 0.2 \text{ cm} \\ \frac{P}{4\eta L} = 500 \text{ dynes/cm}^2 \end{array} \right\} v(r) = 80 - 500 r^2 \text{ cm/s}$$

