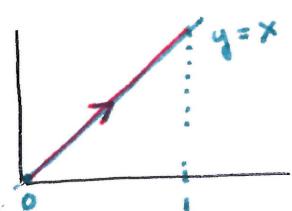


Lecture XXVI § 7.5 Arc Length

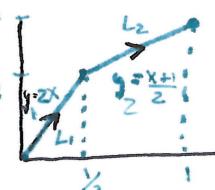
GOAL: Given a curve in the plane, compute its length (\equiv length of a string on top of the curve)

Eg1:



$$L = \sqrt{2} = (y' = 1)$$

Eg2



$$L = L_1 + L_2 = \frac{\sqrt{15}}{3}$$

$$L_1 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$(y'_1 = \frac{1}{2}, y'_2 = \frac{1}{2})$$

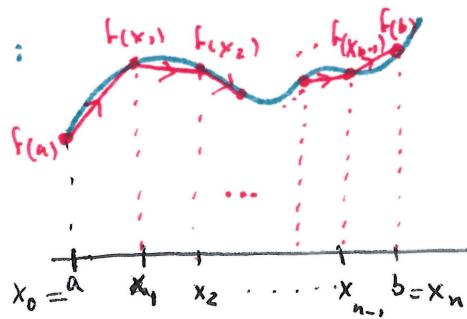
$$L_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

In general: Length of a polygonal curve = sum of lengths of all chords.

For general curves: approximate them by polygons & do limit process.

Thm: The arc length of the graph $y=f(x)$ of a cont. & diff'ble function $f:[a,b] \rightarrow \mathbb{R}$ whose derivative is continuous is $L = \int_a^b \sqrt{1+f'(x)^2} dx$.

Proof:



Step 1: Subdivide the interval $[a, b]$ into n intervals $[x_i, x_{i+1}]$ of length $\Delta x_i = x_{i+1} - x_i$.
(can take them $= \frac{b-a}{n}$ if we want)

Step 2: Draw the polygonal path joining $f(x_i)$ to $f(x_{i+1})$ for $i=0, \dots, n-1$

Step 3: The segment joining $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$ (2 consecutive points) has length $L_i = \sqrt{(\Delta x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$

$$= (\Delta x_i) \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{\Delta x_i} \right)^2}$$

Since f is cont & diff'ble, by the mean value theorem, we know

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x_i} = f'(x_i^*) \quad \text{for some } x_i < x_i^* < x_{i+1}$$



$$\text{Step 3: } L_{(\text{polyg})} = \sum_{i=0}^{n-1} L_i = \sum_{i=0}^{n-1} \left(\sqrt{1 + f'(x_i^*)^2} \Delta x_i \right)$$

$$\text{So } L = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} \sqrt{1 + (f'(x_i^*))^2} \Delta x_i$$

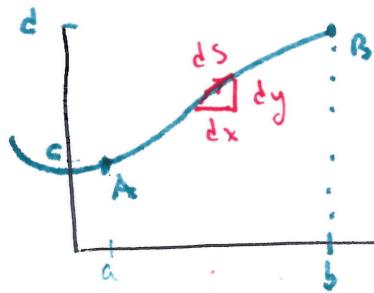
$$= \int_a^b \sqrt{1 + f'(x)^2} dx \quad \text{as long as } \sqrt{1 + f'(x_i^*)^2} \rightarrow \sqrt{1 + f'(x)^2}$$

if $x_i^* \rightarrow x$.

This is guaranteed by the continuity of $f'(x)$. \blacksquare

Note: Often, it is very hard to find the antiderivative of $\sqrt{1 + f(x)^2}$. In these cases, numerical approximation methods are used to find L .

Liberiz Notation:



differential arc length element = ds

$$ds = \sqrt{dx^2 + dy^2}$$

$$= dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = dy \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

$$L_{AB} = \int_A^B ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$$\text{Also } L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy \quad \text{if } y = f(x) \text{ can be inverted } x = x(y).$$

Example 1: Find the length of the curve $y^2 = x^3$ between the points $(0,0)$ & $(4,8)$

(1) Use implicit differentiation: $2y y' = 3x^2$ so $y \neq 0$ we get $y' = \frac{3}{2} \frac{x^2}{y}$

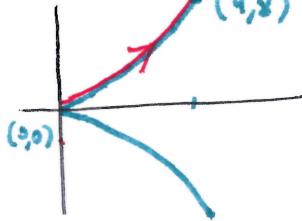
$$L = \int_0^4 \sqrt{1 + y'^2} dx = \int_0^4 \sqrt{1 + \frac{9}{4}x^2} dx = \frac{4}{9} \int_0^{10} \sqrt{u} du \quad \text{so } (y')^2 = \frac{9}{4} \frac{x^4}{y^2} = \frac{9}{4} x^2$$

$$u = 1 + \frac{9}{4}x^2 \quad du = \frac{9}{2}x dx$$

$$x=0 \Rightarrow u=1$$

$$x=4 \Rightarrow u=10$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_0^{10} = \boxed{\frac{80}{27} \sqrt{10}}$$



curve : $cusp$

(2) Solve for y : $y = \sqrt{x^3}$ $y' = \frac{3}{2} \sqrt{x}$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x^2} dx \quad (y')^2 = \frac{9}{4}x$$

int. R₂ [0,4]

Example 2: Circumference of a radius r circle:

$$f(x) = \sqrt{r^2 - x^2} \quad -r \leq x \leq r$$

half-circle.

$$f'(x) = \frac{1}{2} \frac{(-2x)}{\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}} \Rightarrow (f'(x))^2 = \frac{x^2}{r^2 - x^2}$$

$$L = 2 \int_{-r}^r \sqrt{1 + f'^2(x)} dx = 2 \int_{-r}^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 2 \int_{-r}^r \frac{r}{\sqrt{r^2 - x^2}} dx$$

$$= 2 \int_{-1}^1 \frac{r^2}{\sqrt{r^2 - r^2 u^2}} du = 2 \int_{-1}^1 \frac{r^2}{r \sqrt{1-u^2}} du = 2r \int_{-1}^1 \frac{1}{\sqrt{1-u^2}} du$$

$$u = \frac{r}{r} x \quad du = \frac{r}{r} dx \quad = r \underbrace{\left(2 \int_{-1}^1 \frac{1}{\sqrt{1-u^2}} du \right)}_{= \text{circumference of the unit circle.}}$$

We will use trigonometric substitution to solve this integral.

Example 3: $y = \frac{1}{24}x^3 + \frac{2}{x}$ for $2 \leq x \leq 4$

$$y'(x) = \frac{3}{24}x^2 - \frac{2}{x^2} = \frac{1}{8}x^2 - \frac{2}{x^2}$$

$$y'^2 = \frac{x^4}{64} + \frac{4}{x^4} - \frac{4}{8} = \frac{x^4}{64} + \frac{4}{x^4} - \frac{1}{2}$$

$$1 + y'^2 = 1 + \frac{x^4}{64} + \frac{4}{x^4} - \frac{1}{2} = \frac{x^4}{64} + \frac{4}{x^4} + \frac{1}{2} = \left(\frac{1}{8}x^2 + \frac{2}{x^2} \right)^2$$

$$L = \int_2^4 \frac{1}{8}x^2 + \frac{2}{x^2} dx = \frac{1}{24}x^3 + -\frac{2}{x} \Big|_2^4 = \frac{64}{24} - \frac{1}{2} - \left(\frac{8}{24} - 1 \right) = \frac{17}{8}$$