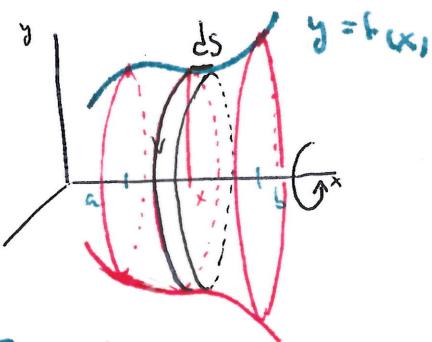


Lecture XXVII : §7.6 The area of a surface of revolution

11

§1 Surfaces of revolution:

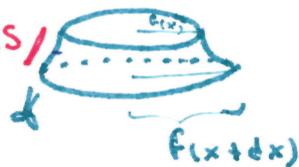


Fix $f: [a, b] \rightarrow \mathbb{R}$ cont & positive & rotate the graph about x-axis
Take the surface of the solid of revolution. We call it the surface of revolution.

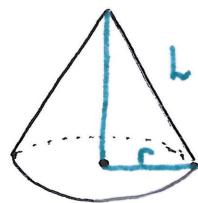
Take strips $[x_i, x_{i+1}]$ & the revolution of the graph f restricted to $[x_i, x_{i+1}]$

Q: What's the surface area of these strips? $\frac{ds}{dx} =$ Frustum (of a cone)

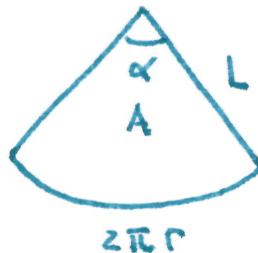
$$dA = 2\pi f(x) ds$$



(1) Model example = cone:



Cut and layout



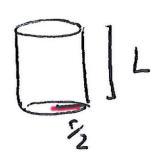
= circular sector of radius L

• Area of circle = πL^2 \Rightarrow area of sector = $\pi L^2 \left[\frac{\alpha}{2\pi} \right]$

• Circumference of sector = $2\pi L \left[\frac{\alpha}{2\pi} \right] = 2\pi r$

So $\frac{\alpha}{2\pi} = \frac{\text{Area sector}}{\text{Area circle}} = \frac{\text{Circumf. sector}}{\text{Circumf. circle}}$ so $\frac{\text{Area sector}}{\pi L^2} = \frac{2\pi r}{2\pi L}$

Conclusion: Area sector = $\pi r L = L \cdot 2\pi \left(\frac{r}{2} \right)$



(2) Area of Frustum:



$$L_1 = L_2 + L$$

$$\begin{aligned} \bullet \text{ Area} &= \text{Area Cirl}(L_1, r_1) - \text{Area Cirl}(L_2, r_2) \\ &= \pi r_1 L_1 - \pi r_2 L_2 \\ &= \pi (r_1 L_1 - r_2 L_2) \end{aligned}$$

$$\bullet \text{ Similarity of } \triangle: \frac{L_1}{r_1} = \frac{L_2}{r_2} \Rightarrow \boxed{r_2 L_1 = r_1 L_2} \quad (*)$$

• We add & subtract $r_1 L_2$:

$$\begin{aligned}
 \text{Area} &= \pi(r_1 L_1 - r_1 L_2 + r_1 L_2 - r_2 L_2) \\
 &\stackrel{\text{use } (1)}{=} \pi(r_1 L_1 - r_1 L_2 + r_2 L_1 - r_2 L_2) = \pi(r_1(L_1 - L_2) + r_2(L_1 - L_2)) \\
 &= \pi(r_1 + r_2)(L_1 - L_2) = \pi(r_1 + r_2)L
 \end{aligned}$$

Inclusion Area Frustum = $\pi(r_1 + r_2)L = 2\pi \left(\frac{r_1 + r_2}{2}\right)L$

Use this to compute surface area of revolution:

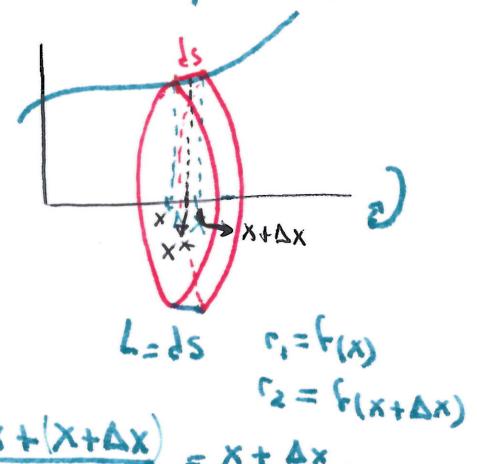
$$\begin{aligned}
 dA &= 2\pi f(x) ds \\
 &= 2\pi f(x) \sqrt{1 + f'(x)^2} dx \quad (\text{lecture XXVI})
 \end{aligned}$$

Note: For this we still need $f'(x)$ continuous!

$$\text{Area of frustum} = 2\pi \left(\frac{f(x) + f(x+\Delta x)}{2} \right) ds$$

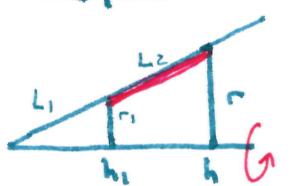
$$\begin{aligned}
 dA &\approx 2\pi \left(\frac{f(x^*)}{2} \right) ds \\
 &\approx 2\pi f(x^*) \sqrt{1 + f'(x^*)^2} dx
 \end{aligned}$$

$$x^* = \frac{x + (x + \Delta x)}{2} = x + \frac{\Delta x}{2}$$



We set Area of revolution = $\int dA = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx$

Example 1: Frustum



$$y = \frac{r}{h}x \Rightarrow y' = \frac{r}{h}$$

$$L = L_1 + L_2$$

$$\text{Similarities: } \frac{h_1}{h} = \frac{r_1}{r} = \frac{L_1}{L}$$

$$h_2 = \sqrt{r^2 + h^2}$$

$$\begin{aligned}
 A &= \int_{h_1}^{h_2} 2\pi \frac{r}{h} x \underbrace{\sqrt{1 + \frac{r^2}{h^2}}}_{= \frac{h^2 + r^2}{h} = \frac{L^2}{h}} dx = 2\pi \frac{r}{h^2} L \int_{h_1}^{h_2} x dx = \pi \frac{r}{h^2} L \left. x^2 \right|_{h_1}^{h_2} \\
 &= \pi \frac{r}{h} \frac{L}{h} (h^2 - h_1^2)
 \end{aligned}$$

$$= \pi \left(rL \left(1 - \frac{h_1^2}{h^2} \right) \right) = \pi rL \left(1 - \frac{r_1^2}{r^2} \right) = \pi \frac{L}{r} (r^2 - r_1^2) = \pi L (r_1 + r) \left(\frac{r - r_1}{r} \right)$$

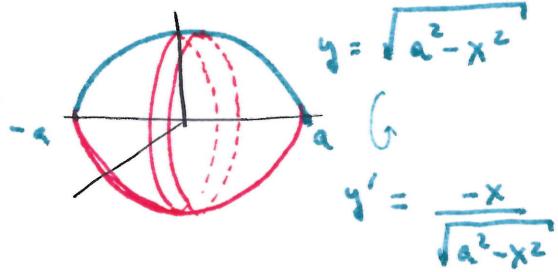
$$= \pi (r_1 + r) L \left(1 - \frac{r_1}{r} \right) = \pi (r_1 + r) L \left(1 - \frac{L_1}{L} \right) = \pi (r_1 + r) (L - L_1)$$

$$= \boxed{\pi (r_1 + r) L_2}$$

as we expected from our previous formula!

Example 2: Cone: $r_1 = h_1 = b_1 = 0$ $A = \pi r L$

Example 3 Sphere of radius a .



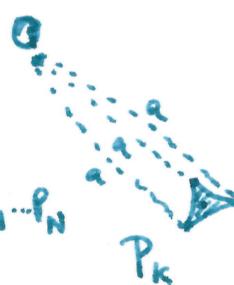
$$\begin{aligned}
 A &= \int_{-a}^a 2\pi \sqrt{a^2 - x^2} \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx \\
 &= \int_{-a}^a 2\pi \sqrt{a^2 - x^2} \sqrt{\frac{a^2}{a^2 - x^2}} dx \\
 &= 2\pi a \int_{-a}^a dx = 2\pi a \times \left[x \right]_{-a}^a = 4\pi a^2
 \end{aligned}$$

Alternative proof (Archimedes)

Cover the surface with "triangles" & draw the pyramids with vertex at the center of the sphere.

Surface of sphere gets.

Triangulated $2N$ pyramids P_1, P_2, \dots, P_N cover the sphere.



$$\text{Vol Pyramid } P_k = \frac{1}{3} a A_k.$$

height $\approx a$

A_k = area of base



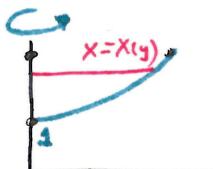
$$\sum_{k=1}^N \text{Vol}(P_k) = \sum_{k=1}^N \frac{1}{3} a A_k = \frac{1}{3} \left(\sum_{k=1}^N A_k \right) a$$

$$\text{So } \lim_{N \rightarrow \infty} \sum_{k=1}^N \text{Vol}(P_k) = \text{Vol Sphere} = \frac{4}{3} \pi a^3 \quad \left. \right\}$$

$$\lim_{N \rightarrow \infty} \frac{1}{3} \left(\sum_{k=1}^N A_k \right) a = \frac{1}{3} a \text{ Area Sphere}$$

$$\left. \begin{aligned} \text{get Area} &= \frac{\frac{4}{3} \pi a^3}{a} \\ \text{so Area} &= 4 \pi a^2 \end{aligned} \right\}$$

Example 4: $y = \frac{2}{3} (1+x^2)^{3/2}$ $0 \leq x \leq 3$ rotate about the y-axis



Solve for x :

$$\frac{3}{2} y = (1+x^2)^{3/2}$$

$$\left(\frac{3}{2} y \right)^{2/3} - 1 = x^2$$

$$\text{and } \sqrt{\left(\frac{3}{2} y \right)^{2/3} - 1} = x$$

$$\begin{aligned}
 A &= \int_{\frac{2}{3}(10)^{3/2}}^{\frac{2}{3}(10)^{3/2}} 2\pi \sqrt{\left(\frac{3}{2} y \right)^{2/3} - 1} \sqrt{1 + \frac{1}{4\left(\frac{3}{2}y\right)^{2/3}\left(\left(\frac{3}{2}y\right)^{2/3}-1\right)}} dy \\
 &= \frac{1}{2} \int_1^{2\pi} 2\pi \sqrt{4\left(\frac{3}{2}y\right)^{2/3} \left(\left(\frac{3}{2}y\right)^{2/3}-1\right) + 1} dy
 \end{aligned}$$

$$x' = \frac{1}{2} \frac{\left(\frac{3}{2}y\right)^{-1/3}}{\sqrt{\left(\frac{3}{2}y\right)^{2/3}-1}}$$

$$\text{Substitution : } u = \left(\frac{3}{2}y\right)^{\frac{2}{3}} \quad du = \frac{2}{3} \cdot \frac{3}{2} \left(\frac{3}{2}y\right)^{-\frac{1}{3}} dy = \frac{dy}{\left(\frac{3}{2}y\right)^{\frac{1}{3}}} \quad [4]$$

$$y=1 \rightarrow u = \left(\frac{3}{2}\right)^{\frac{2}{3}}$$

$$y = \frac{2}{3}10^{\frac{3}{2}} \Rightarrow u = 10.$$

$$\begin{aligned} A &= \int_{\left(\frac{3}{2}\right)^{\frac{2}{3}}}^{10} \pi \sqrt{4u(u-1)+1} du = \int_{\left(\frac{3}{2}\right)^{\frac{2}{3}}}^{10} \pi \sqrt{4u^2 - 4u + 1} du \\ &= \int_{\left(\frac{3}{2}\right)^{\frac{2}{3}}}^{10} \pi (2u-1) du = \pi (u^2 - u) \Big|_{\left(\frac{3}{2}\right)^{\frac{2}{3}}}^{10} = \pi \left(90 - \frac{3}{2} \sqrt{\frac{3}{2}} + \left(\frac{3}{2}\right)^{\frac{2}{3}} \right) \end{aligned}$$