

# Lecture XXVIII : §7.7 Work and Energy

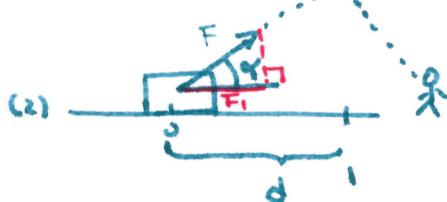
## §1 Work :

"Effect done by a force to move an object; force must act in the direction of movement"



F constant (pulling an object with a chord)

$$W = F \cdot d \quad F = \text{force} \quad d = \text{distance}$$



comp of F in the direction of movement:  $F_i = F \cos \alpha$

$$W = (F \cos \alpha) d.$$

Units:  $F = m a \rightarrow \text{kg} \cdot \frac{m}{s^2} = (\text{mass}) \frac{(\text{distance})^2}{(\text{time})^2}$

$$W = F d \rightarrow \text{kg} \frac{m^2}{s^2} = \left( \text{kg} \frac{m}{s^2} \right) (\text{distance})$$

Eg: Weight = measure of force with which an object is attracted to earth.

$$\Rightarrow W = \text{ft-pounds} / \boxed{\text{ft-lbs}}.$$

- If we lift a box weighing 20 lbs 3 ft high, the work done is 60 ft-lbs.

Other units: (1) cgs (centi-gram-second) system:  $F$  in dynes  $= 1 \frac{\text{cm}}{\text{s}^2}$  g.

(2) mks (m-kg-second) system:  $F$  in Newtons  $= 1 \frac{\text{m}}{\text{s}^2}$  kg.

$$\Rightarrow W \text{ in cgs} = \text{dyne-cm} = \text{erg} \quad W \text{ in mks} = \text{N-m} = \text{Joule}$$

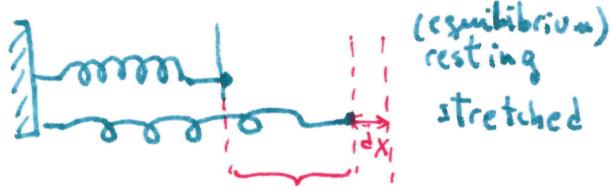
Conversions:  $1 \text{ ft-lb} \cong 1.356 \text{ J} \quad 1 \text{ J} = 10^7 \text{ ergs.}$

Q: What if the force is non-constant?  $\rightarrow F = F(x)$  changes with  $x = \text{distance}$

Over a small distance  $\Delta x$ , the work is  $\Delta W = F \cdot \Delta x$  ("like  $F$  where constant")

$$\text{So } dW = F dx \xrightarrow[\text{FTC}]{\text{ }} W_{(u)} = \int_0^u F(x) dx \quad \text{for } u \in \mathbb{R} \text{ (distance)}$$

• Prototypical example: Springs  $\rightarrow$  force pushes back to equilibrium & more force acts the further we stretch the spring



$\leftarrow F$  restorative force.

(against the movement of stretching)

Hooke's Law

$$F(x) = kx$$

$k = \text{spring constant}$  (depends on material properties of spring)

Ex: Assume it takes 8 lb of force to hold a 16 in long spring 2 in away.  
How much work is done when moving the spring to a 24 in length?

Sol: ① Use state to find  $k$ .

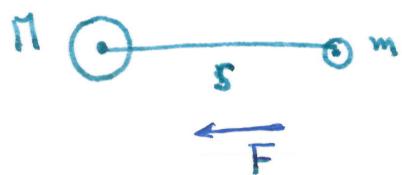
$$F(2) = k \cdot 2 \text{ in} = 8 \text{ lb} \Rightarrow k = 4 \frac{\text{lb}}{\text{in}}$$

②  $x = \text{distance} = 24 - 16 = 8$

$$W = \left( \int_0^8 4 \times dx \right) \frac{\text{lb}}{\text{in}} = 2x^2 \Big|_0^8 \frac{\text{lb}}{\text{in}} = 128 \text{ in}^2 \frac{\text{lb}}{\text{in}} = 128 \text{ in-lb}$$

• Prototypical example 2: Gravity

Recall: Force of attraction of 2 bodies of masses  $M$  &  $m$ . (replace bodies by a pt of mass  $M, m$ )



$$F(s) = -G \frac{Mm}{s^2}$$

$G^0$  = universal gravity constant

(gravity of  $M \approx m$ )

(Eq.  $M$  for earth,  $m$  = rocket)

$$G = g R^2 \quad \text{see Lect XIX}$$

$$\text{So } dW = -G \frac{Mm}{s^2} ds$$

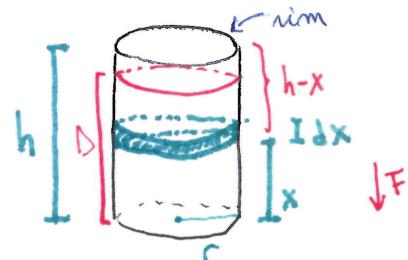
against the force of gravity

To move  $m$  ( $\propto M$ ) from  $s=a$  to  $s=b$  requires:

$$W = - \int_a^b F(s) ds = +GMm \int_a^b \frac{ds}{s^2} = -GMm \left( \frac{1}{b} - \frac{1}{a} \right) \begin{cases} > 0 \text{ if } b > a \\ < 0 \text{ if } b < a. \end{cases}$$

Remark: If  $b \rightarrow \infty$  then  $W \xrightarrow{b \rightarrow \infty} \frac{GMm}{a} > 0$  is potential of the two particle  
(work required to separate them completely!)

• Prototypical example 3: Pumping water from a tank



• Cylindrical tank

$$\begin{cases} \text{radius} = r \\ \text{height} = h \end{cases}$$

• Filling with water to depth D

Body = drops of water

$$\begin{aligned} W &= \text{weight} \cdot \text{density of water} \\ &= \text{weight} / \text{unit volume} = \frac{\text{lb}}{\text{in}^3} \end{aligned}$$

Q: How much work is done to pump the water out of the rim of the tank?

Key Assumption: Work is the same for all drops of water at the same distance below the rim.

Idea: If we have a small volume   $\Delta V = \pi r^2 dx$ , the weight is  $F_{(x)} = \text{density} \cdot \Delta V = w \pi r^2 dx$  (distance to earth is "the same" at any height)

Since we are moving out, the distance travelled is  $h-x$ .

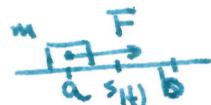
$$dW = F_{(x)} (h-x) = w \pi r^2 (h-x) dx.$$

$$\text{So } W = \int_0^D w \pi r^2 (h-x) dx = w \pi r^2 \left( hx - \frac{x^2}{2} \right) \Big|_0^D \\ = w \pi r^2 D \left( h - \frac{D}{2} \right)$$

Note: If we measure from the top:

$$W = \int_{h-D}^h w \pi r^2 x dx = w \pi r^2 \frac{x^2}{2} \Big|_{h-D}^h = \frac{w \pi r^2}{2} (h^2 - (h-D)^2) = w \pi r^2 D \left( \frac{h-D}{2} \right)$$

§2 Energy:



$$F = m \frac{dv}{dt}$$

$$v = \frac{ds}{dt} \quad \text{velocity}$$

$$\text{Kinetic energy} = \text{energy due to motion} = \frac{1}{2} m v^2$$

Theorem: The work done by  $F$  equals the change in the kinetic energy of the particle.

Proof: Rewrite  $F = m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} = m v \frac{dv}{ds}$   
via chain rule

$$W = \int_a^b F(s) ds = \int_a^b m v \frac{dv}{ds} ds = \int_{s(a)}^{s(b)} m v \ dv = \frac{1}{2} m v^2 \Big|_{s(a)}^{s(b)} \\ = \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2 \quad \square$$

[Assumption:  $F$  depends only on  $s$ .]

We define  $V(s) = - \int F(s) ds$  potential energy

$$\text{Then } W = -(V(s) - V(a)) \underset{(s \rightarrow s)}{=} V(a) - V(s)$$

$$\text{By Thm: } V(a) - V(s) = \frac{1}{2} m v(s)^2 - \frac{1}{2} m v_a^2$$

$$E := V(a) + \frac{1}{2} m v_a^2 = \frac{1}{2} m v^2 + V(s)$$

↑  
indep of  $s(t)$ .

TOTAL energy of the particle       $\begin{matrix} \text{kinetic} \\ \text{energy} \end{matrix}$        $\begin{matrix} \text{potential} \\ \text{energy} \end{matrix}$

Nice Application: Working heart       $W = 0.74 \text{ ft-lb. of a single stroke.}$

Law of Conservation  
of Energy