

Lecture XXIX § 8.2 Exponents & Logarithms

§ 1 Basics on exponents

$a > 0$, $x \in \mathbb{R}$ \rightsquigarrow Q = What is a^x ?

- Example ① If $x = n$ integer then:

- If $n > 0$: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$

- If $n = 0$: $a^0 = 1$ (by def.)

- If $n < 0$: $a^n = (a^{-1})^{-n} = \underbrace{\frac{1}{a} \cdot \frac{1}{a} \cdots \frac{1}{a}}_{n \text{ times}}$

- ② If $x = \frac{p}{q}$ rational ($p, q \in \mathbb{Z}$, $q \neq 0$, $\gcd(p, q) = 1$, $q > 0$)

$$a^{\frac{p}{q}} = (\sqrt[q]{a})^p \quad \text{where } \sqrt[q]{a} \text{ is the unique positive number st } (\sqrt[q]{a})^q = a$$

(positive root of $x^q - a = 0$)

AIM: Define a^x via a limiting process using rational approximation

Properties from ① & ②:

$$(1) a^m a^n = a^{m+n}, \quad (2) \frac{a^m}{a^n} = a^{m-n}, \quad (3) (a^m)^n = a^{mn}$$

- clear for m, n integers.

• Remark: $a^{\frac{p}{q}} = a^{\frac{kp}{kq}}$ for any $k > 0$ integer (Proof: $a^{\frac{kp}{kq}} = (\sqrt[kq]{a})^{kp} = ((\sqrt[q]{a})^k)^{kp}$)

↓
- by uniqueness
 $\sqrt[kq]{a} = \sqrt[q]{a^k}$
* Prop (3)

• To prove (1) - (3) for rational exponents, we can take common denominators, i.e.

$$m = \frac{p}{q}, \quad n = \frac{r}{s}$$

$$(1) a^{\frac{p}{q}} a^{\frac{r}{s}} = (\sqrt[q]{a})^p (\sqrt[s]{a})^r = (\sqrt[q]{a})^{p+r} = a^{\frac{p+r}{q+s}}$$

def b in R (1) to integers

$$(2) \frac{a^{\frac{p}{q}}}{a^{\frac{r}{s}}} = \frac{(\sqrt[q]{a})^p}{(\sqrt[s]{a})^r} = (\sqrt[q]{a})^{p-r} = a^{\frac{p-r}{q-s}}$$

def (2) pos int exponents

$$(3) (a^{\frac{p}{q}})^{\frac{r}{s}} = (\sqrt[q]{a})^p)^{\frac{r}{s}} \stackrel{\text{uniqueness}}{=} ((\sqrt[q]{a})^p)^r \stackrel{\text{by (3)}}{=} (\sqrt[q]{a})^{pr} = a^{\frac{pr}{q^s}}$$

Def: $a^x = \lim_{\substack{r \rightarrow x \\ r \in \mathbb{Q}}} a^r$. for any x in \mathbb{R}

Elementary but tedious to prove:

- This definition extends the construction of $a^{\frac{p}{q}}$. (can take the constant approximation $r = \frac{p}{q}$, but why do other $r \rightarrow \frac{p}{q}$ give the same answer?)
- Properties (1) — (3) hold for a^x as well.
- The function is continuous

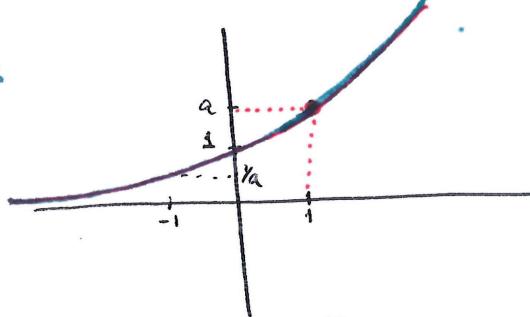
⇒ General exponential functions: $y = a^x$ for x in \mathbb{R} , $a > 0$ fixed

- $a = 1$: constant function $y = 1$.
- $a > 1$: $y = a^x$ is continuous by construction

- y is strictly increasing: $x < y$ then $a^x < a^y$.

$$\lim_{x \rightarrow -\infty} a^x = 0 \quad \lim_{x \rightarrow \infty} a^x = +\infty$$

- graph:



- true for integers ✓
- $(\sqrt[3]{a})^p < (\sqrt[3]{a})^r$ so true for rationals ✓
- limiting process gives \mathbb{R})

⇒ Range = $\{y > 0\}$

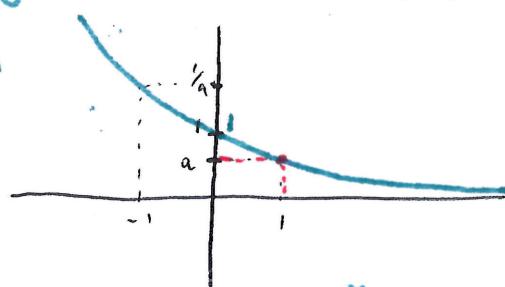
- injective b/c str. increasing

- $a \leq 1$ we set $y = a^x = (\frac{1}{a})^{-x}$ & $\frac{1}{a} > 1$. ⇒ graph & properties get reflected

- y is str. decreasing

$$\lim_{x \rightarrow -\infty} a^x = +\infty \quad \lim_{x \rightarrow \infty} a^x = 0.$$

- graph



⇒ Range = $\{y > 0\}$

- injective b/c str. decreasing

Remark: From the graphs, a^x seems to be smooth and moreover, infinitely differentiable.

- $y = a^x$ ($a \neq 1$) is injective & Range = $\{y > 1\}$, so we have an inverse function

S2 Logarithms Define $y: (0, +\infty) \rightarrow \mathbb{R}$ $\log_a x = y$ if $x = a^y$,
 (Logarithm of x to the base a)

Basic properties (inherited from a^x)

$$(1) \log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$$

Why? $y_1 = \log_a x_1, y_2 = \log_a x_2 \Rightarrow x_1 = a^{y_1}, x_2 = a^{y_2} \quad (1)$
 $x_1 x_2 = a^{y_1} a^{y_2} \Rightarrow x_1 x_2 = a^{y_1+y_2} \Rightarrow \log_a(x_1 x_2) = y_1 + y_2$

$$(2) \log_a(\frac{x_1}{x_2}) = \log_a x_1 - \log_a x_2$$

Why? $y_1 = \log_a x_1, y_2 = \log_a x_2 \Rightarrow x_1 = a^{y_1}, x_2 = a^{y_2} \quad (2)$
 $\frac{x_1}{x_2} = \frac{a^{y_1}}{a^{y_2}} = a^{y_1-y_2} \Rightarrow \log_a(\frac{x_1}{x_2}) = y_1 - y_2$

$$(3) \log_a(x^b) = b \log_a x$$

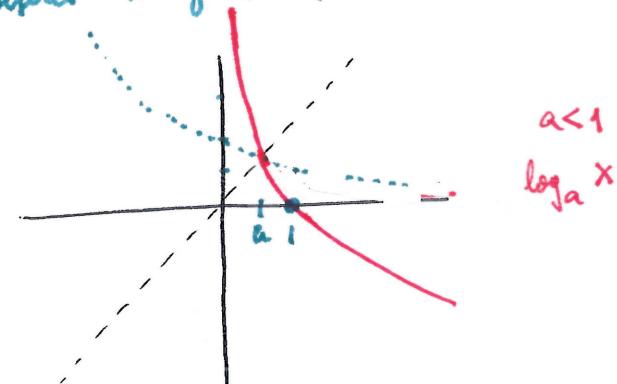
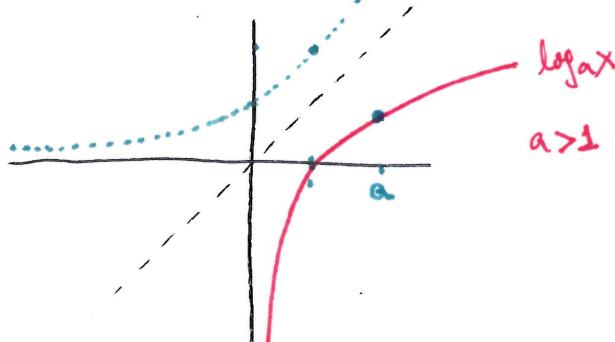
Why? $y = \log_a x \Rightarrow x = a^y \Rightarrow x^b = (a^y)^b = a^{by} \Rightarrow \log_a(x^b) = by$

$$(4) \log_a(a^x) = x \quad \text{by def } (y = \log_a a^x \text{ then } a^y = a^x \text{ but by injectivity } y = x)$$

$$(5) a^{\log_a x} = x \quad \text{by def } (y = \log_a x \Rightarrow a^y = x)$$

$$(6) \log_a a = 1, \log_a 1 = 0 \quad (a^0 = 1)$$

(7) Graph: Turn over the figure for a^x about $x=y$ line.



$$\log_a x = y \Rightarrow x = a^y = (\frac{1}{a})^{-y}$$

$$y = \log_{1/a} x$$

\Rightarrow flip the graph for $a > 1$ about the x-axis

Note: If $a > 1$: $\lim_{x \rightarrow 0^-} \log_a x = -\infty$

$$\lim_{x \rightarrow \infty} \log_a x = +\infty$$

If $a < 1$ $\lim_{x \rightarrow 0^-} \log_a x = +\infty$

$$\lim_{x \rightarrow \infty} \log_a x = -\infty$$

Next time: Particular bases for log with nice properties.