

Lecture XXX: §8.4 The natural logarithm function $y = \ln x$
 §8.3 The Number e and the function $y = e^x$.

§1 Derivatives of exponentials a^x :

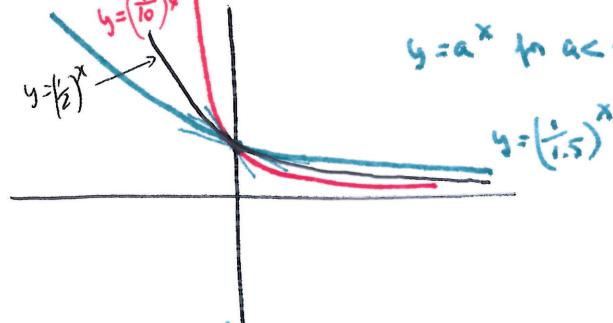
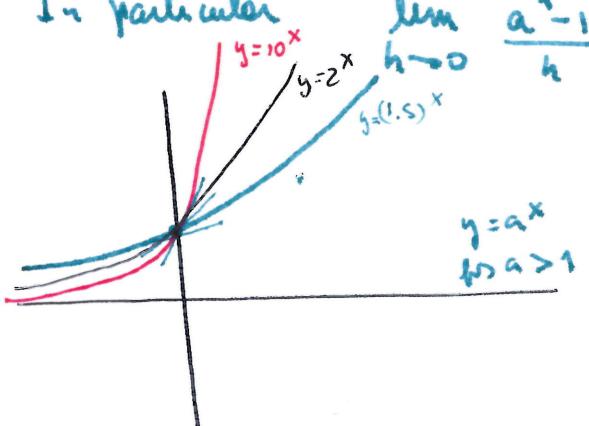
Last time: We claimed $y = a^x$, $a > 0$, $a \neq 1$ was differentiable.

Q: What's y' ? Use increments

$$\frac{d}{dx} a^x = \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

as long as $\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$ exists!

In particular $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \left. \frac{d}{dx} a^x \right|_{x=0}$ = slope of the tangent line to a^x at $(0, 1)$



Def.: e is the real number satisfying $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$ (slope = 1)!

Consequence: $y = e^x$ has the property that $\boxed{\frac{d}{dx} e^x = e^x}$ (the exponential function!)

For any constant $c \in \mathbb{R}$ $\frac{d}{dx}(ce^x) = c \frac{d}{dx} e^x = ce^x$.

So the equation $y' = y$ is satisfied by $y = ce^x$ for any $c \in \mathbb{R}$. (1-parameter family of solns!)

Prop: These are all the solutions!

Proof: Pick a solution $f(x)$ & take $\frac{d}{dx}\left(\frac{f}{e^x}\right) = \frac{f'e^x - f(e^x)}{(e^x)^2} = \frac{fe^x - fe^x}{(e^x)^2} = 0$.
 So $\frac{f'(x)}{e^x} = c$ for some constant c !

→ One more function to compute derivatives & integrals!

$$\text{Eg: (1) } \frac{d}{dx} e^{x^2} = e^{x^2} \cdot \frac{d}{dx}(x^2) = e^{x^2} \cdot 2x, \quad (2) \frac{d}{dx} e^{-x} = -e^{-x}$$

chain rule

$$(3) \frac{d}{dx} e^{x^2} = -e^{-x} \cdot x^2, \text{ etc.}$$

Prop $\frac{d}{dx} e^x = e^x$ so $\int e^x dx = e^x + C$ \mapsto any constant C

$$\text{Eg } \int e^{5x} dx = \int e^u \frac{du}{5} = \frac{1}{5} (e^u + C) = \frac{1}{5} (e^{5x} + C) = \frac{e^{5x}}{5} + \tilde{C}.$$

$u=5x$
 $du=5dx$

$$\int x e^{x^2} dx = \int \frac{x}{2} e^{x^2} dx = \frac{1}{2} \int e^u du = \frac{e^{x^2}}{2} + C.$$

$u=x^2$

32 The exponent e:

① Q: How big is e?

$$\text{Starting point } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$\frac{e^h - 1}{h} - 1 = \frac{e^h - 1}{h} - \frac{h}{h} = \frac{e^h - 1 - h}{h}$$

so if h is small $\frac{e^h - 1 - h}{h} \approx 0$
 $e^h \approx 1 + h$
 $e \approx (1+h)^{1/h}$

$$\text{In particular } e = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = \lim_{n \rightarrow \infty} (1+\frac{1}{n})^n \quad (*)$$

Consequence: We can numerically approximate e letting n be integer.

$$e \approx 2.718281828459045 \dots$$

② FACT: e is irrational & transcendental, just like π (e cannot be a root of a polynomial with rational coefficients)

③ Q: How fast does e^x grow?

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = +\infty \text{ for any } n \text{ integer} \quad (\text{Equivalently: } \lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0)$$

So e^x grows faster than ANY polynomial!

$$\text{(i) Proof of: } e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad [\text{Appendix A.7}]$$

Start with binomial theorem: For n integer, define $X_n := \left(1 + \frac{1}{n}\right)^n$

$$X_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \dots + \frac{n!}{n! n^n} \quad (n+1 \text{ terms})$$

$$= 1 + 1 + \frac{1}{1 \cdot 2} \left(\frac{n-1}{n}\right) + \frac{1}{1 \cdot 2 \cdot 3} \frac{n-1}{n} \cdot \frac{n-2}{n} + \dots + \frac{1}{n!} \frac{(n-1)(n-2)\dots(n-n)}{n^n}$$

$$= 1 + 1 + \frac{1}{1 \cdot 2} \left(1 - \frac{1}{n}\right) + \frac{1}{1 \cdot 2 \cdot 3} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots + \frac{1}{n!} \left(\frac{1-1}{n}\right) \left(\frac{1-2}{n}\right) \dots \left(\frac{1-(n-1)}{n}\right)$$

As n increases: number of terms in x_n grows

- value of each term decreases

Conclusion: $x_1 < x_2 < x_3 < \dots < x_n < x_{n+1} < \dots$

Also: $0 < 1 - \frac{k}{n} < 1$ for all $1 \leq k \leq n$ so $x_n < 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots + \frac{1}{1 \cdot 2 \cdot 3 \cdots n}$

Now: $x_n < 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots + \frac{1}{n!} < 1 + 1 + \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \right) < 3$

Conclusion: (x_n) steadily increases & bounded above by 3 so they have a limit

So $\lim_{n \rightarrow \infty} x_n = \lim_{\substack{n \rightarrow \infty \\ n \text{ integer}}} \left(1 + \frac{1}{n}\right)^n$ exists! ($= e$ by definition)

• Next show $\lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = e$.

• Pick $0 < h < 1$, so we can find n integer with $n \leq \frac{1}{h} < n+1$.

$$\text{Now } 1 + \frac{1}{n+1} < 1 + h < 1 + \frac{1}{n}$$

$$\left(1 + \frac{1}{n+1}\right)^n < (1+h)^{\frac{1}{h}} < \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\begin{aligned} \left(\frac{1}{1 + \frac{1}{n+1}}\right) \cdot \left(1 + \frac{1}{n+1}\right)^{n+1} &< (1+h)^{\frac{1}{h}} < \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) \\ \downarrow & \quad \downarrow n \rightarrow \infty & \downarrow n \rightarrow \infty & \downarrow n \rightarrow \infty & \text{As } h \rightarrow 0^+ \\ \frac{1}{1+h} &= 1 & e & & n \rightarrow \infty \end{aligned}$$

By squeeze lemma $(1+h)^{\frac{1}{h}} \xrightarrow[h \rightarrow 0^+]{ } e$

• For $h \rightarrow 0^-$: write $(1+h)^{\frac{1}{h}} = (1-(-h))^{\frac{1}{-h}} = \left(\frac{1}{1-k}\right)^{\frac{1}{k}} = \left(1 + \frac{k}{1-k}\right)^{\frac{1}{k}}$

$$\begin{aligned} &= \left(1 + \frac{k}{1-k}\right)^{\frac{1-k}{k}} \left(1 + \frac{k}{1-k}\right) \xrightarrow[h \rightarrow 0^- \quad (k \rightarrow 0^+)]{} e \cdot 1 = e. \quad \blacksquare \\ &k = \frac{1-k}{k} + 1 \end{aligned}$$

(Corollary: If $a = e^b$ for some b , then $\frac{a^h - 1}{h} = \frac{e^{bh} - 1}{bh} = b \left(\frac{e^{bh} - 1}{bh} \right) \xrightarrow[h \rightarrow 0]{} b$

So $\frac{d}{dx} a^x = ba^x$ & a^x is infinitely differentiable!

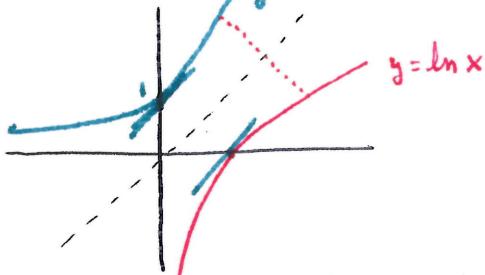
§ 3 The natural logarithm:

Def.: $\ln(x) = \log_e(x)$ so $y = \ln x$ means $e^y = x$.

Prop: $\ln(x)$ is infinitely differentiable & $\frac{d}{dx} \ln x = \frac{1}{x}$

Proof: Use implicit differentiation:

$$x = e^y \text{ so } \frac{d}{dx} \text{ gives } 1 = e^y y' \text{ so } y' = \frac{1}{e^y} = \frac{1}{x} \quad \blacksquare$$



- Slopes of tangent lines for e^x & $\ln x$ are related!
- $\lim_{x \rightarrow 0^+} \ln x = -\infty$, $\lim_{x \rightarrow \infty} \ln x = +\infty$
(because $e > 1$)

• Powerful integration rule!
 $(\int x^{-1} dx = \ln|x| + C)$

• In general $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}$ so $\boxed{d \ln u = \frac{1}{u} du}$

$$\int \frac{dx}{x} = \ln|x| + C \quad (\text{if } x < 0: \int \frac{dx}{x} = \int \frac{dx}{-x} = \ln(-x) + C)$$

$$\text{Eg: } \frac{d}{dx} \ln(x^2) = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \quad (\text{Note: } \ln(x^c) = c \ln x \text{ so } \frac{d}{dx} \ln x^c = c \frac{1}{x})$$

$$\begin{aligned} \frac{d}{dx} \ln(\sqrt{x^2+1}) &= \frac{d}{dx} (\ln x + \ln \sqrt{x^2+1}) = \frac{d}{dx} (\ln x + \frac{1}{2} \ln(x^2+1)) \\ &= \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} = \frac{1}{x} + \frac{x}{x^2+1}. \end{aligned}$$

$$\cdot \int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\log |\cos x| + C \quad \text{for } x > 0$$

$$\cdot \int \frac{x^3}{x^4+1} dx = \frac{1}{4} \int \frac{du}{u} = \frac{1}{4} \ln(u^4+1) + C. \quad \begin{matrix} u = \cos x \\ u = x^4+1 \end{matrix}$$

Application: $y' = ky$ has solutions: $y = ce^{kx}$ for c constant
(1-parameter family of solns)

Why? Separation of variables $\frac{y'}{y} = k \Rightarrow \ln y = \int \frac{dy}{y} = \int k dx = kx + C'$

$$\text{Take } e^{\square}: y = e^{\ln y} = e^{kx+C'} = \boxed{e^C} e^{kx}$$

$$\begin{aligned} \text{Other solutions? } \left(\frac{y}{e^{kx}} \right)' &= \frac{y' e^{kx} - y(k \cdot e^{kx})}{e^{2kx}} = \frac{ky e^{kx} - ky e^{kx}}{e^{2kx}} = 0. \\ \text{so } y &= ce^{kx} \text{ for some } c \text{ constant.} \end{aligned}$$