

Lecture XXXI: §8.4 The natural logarithm function
 §8.5 Population growth & radioactive decay.

§1 Other bases of logarithm:

Recall: $y = e^x \Rightarrow \frac{dy}{dx} = e^x$

If $a > 0$, then $a = e^{\log_e(a)}$ so $a^x = e^{x \log_e(a)}$

So $\frac{d a^x}{dx} = e^{x \log_e a} \log_e a = \boxed{\ln a a^x}$

In particular $\boxed{\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \left. \frac{d a^x}{dx} \right|_{x=0} = \ln a}$

Similarly: $\boxed{y = \log_a x}$ is equiv to $x = a^y = e^{y \ln a}$ so $\ln x = y \ln a$

$\boxed{y = \frac{\ln x}{\ln a}}$

In particular $\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$

Q Growth? $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$
 (Problem 13 §8.4)

Furthermore: $\lim_{x \rightarrow \infty} \frac{\ln x}{x^p} = 0$ for all $p > 0$

§2 Population growth:

Basic model: $N(t)$ = population at time t (eg bacteria)

Assumptions: unlimited food, no predators, no deaths.

Rate of change of current population: $\boxed{\frac{dN}{dt} = k N(t)}$ for some k constant

Separation of variables: $\frac{dN}{N} = k dt$ + integration gives $k = \frac{N'}{N} = \%$ pop. increase

$\ln N = kt + C'$ so $N = e^{C'+kt}$

$N(t) = \boxed{e^C} e^{kt}$

Usually: initial conditions $N(0) = N_0 > 0$ "C > 0"

gives $\boxed{N(t) = N_0 e^{kt}}$

N_0 = population at time $t=0$

k = rate of increase as % of population

• k vs doubling time

$t_d = \underline{\text{Doubling time}}$ = time it takes for the population to double. Assume $N_0 > 0$

$2N_0 = N(t) = N_0 e^{kt_d}$

so $2 = e^{kt_d}$

so $\boxed{kt_d = \ln 2}$

OR $\boxed{t_d = \frac{\ln 2}{k}}$
 OR $k = \frac{\ln 2}{t_d}$

Remark: $N(t+t_d) = N_0 e^{k(t+t_d)} = N_0 e^{kt} e^{kt_d} = N(t) e^{kt_d} = 2N(t)$ $(kt_d) = \ln 2$

So it doesn't matter when we start! The doubling time is the same!

Eg: # bacteria in a culture doubles every hour. How long does it take for 1000 bacteria to produce 1 billion = 10^9 ?

Soln $t_d = 1$ hour so $k = \frac{\ln 2}{t_d} = \ln 2$ $N(t) = N_0 e^{kt}$
 $N_0 = 1000$ $N(t) = 10^9 e^{(\ln 2)t}$ gives $10^6 = e^{\ln 2 t}$

$6 \ln 10 = (\ln 2) t$ so $t = \frac{6 \ln 10}{\ln 2} = 6 \log_2 10$

Eg 2: In 1970 the world population was 3.6 billion. The Earth weighs $6586 \cdot 10^{18}$ tons. If the population increases at a rate of 2% / year & an average person weighs 120 lb, when will the weight of all people equal the weight of earth?

Sol: $k = \frac{2}{100}$ $N_0 = 3.6 \cdot 10^9$ $\Rightarrow N(t) = 3.6 \cdot 10^9 e^{\frac{2}{100}t}$

$120 \cdot N(t) \geq 6586 \cdot 10^{18} \cdot 2000$ (1 ton = 2000 lb)

$120 \cdot 3.6 \cdot 10^9 e^{\frac{t}{50}} \geq 6586 \cdot 10^{21} \cdot 2$

$e^{t/50} = \frac{6586 \cdot 2 \cdot 10^{21-9}}{12 \cdot 36} = \frac{6586 \cdot 2}{216} \cdot 10^{12} = \frac{3293}{108} \cdot 10^{12}$

$t = 50 \ln \frac{3293}{108} \cdot 10^{12} = 50 (\ln 3293 - \ln 108 + 12 \ln 10) = 1552.92$ years.

§3 Radioactive decay

Characteristic feature of radioactive materials: instead of growth, we have decay so we write $\frac{dx}{dt} = -kx$ for some $k > 0$ constant (called decay constant)

• Solution: $x(t) = C e^{-kt}$ and $C = x(0)$ = amount of material at time $t=0$ (initial material)

Note: Since $e^{-kt} \neq 0$ for all t , radioactive materials NEVER completely decay (at least not in this model!)

• Analogy of doubling time is the half-time $t_{1/2}$ = time it takes for the substance to decay to half its original amount. $x(t_{1/2}) = \frac{1}{2} x(0)$

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So $x(t_{1/2}) = x_0 e^{-kt_{1/2}} = \frac{1}{2} x_0$ so by $\ln \rightarrow -kt_{1/2} = \ln \frac{1}{2} = -\ln 2$

$kt_{1/2} = \ln 2$, $t_{1/2} = \frac{\ln 2}{k}$ OR, $k = \frac{\ln 2}{t_{1/2}}$

Note: As with growth, the initial time is irrelevant!

$x(t+t_{1/2}) = x_0 e^{-k(t+t_{1/2})} = x_0 e^{-kt} e^{-kt_{1/2}} = \frac{1}{2} x_0 e^{-kt} = \frac{1}{2} x(t)$

Example: Cesium 137 decays to 20% in 10 years. What is its half-life?

$x_0 e^{-k \cdot 10} = x(10) = \frac{20}{100} x_0 = \frac{1}{5} x_0$ so $e^{-k \cdot 10} = \frac{1}{5}$
 $-k \cdot 10 = -\ln 5$

So $\frac{1}{2}$ life is $t_{1/2} = \frac{\ln 2}{\frac{\ln 5}{10}} = \frac{10 \ln 2}{\ln 5} = 10 \log_5 2$ years.
 $= 9.3067$ years.

Application: Radiocarbon dating (Libby's 1940s) Carbon 14 in a living thing starts decaying right after its death

Carbon 14: radio carbon & half-time life ~ 5600 years

Piece of old wood has $\frac{1}{2}$ radioactivity from Carbon 14 as a living tree had, then it lived about 5600 years ago. If it has $\frac{1}{4}$ of radioactivity, then it lived 11,200 years.
 \rightarrow Verified on sequoia trees, furniture from Egyptian tombs whose age is known by other means.

Example: Cobalt 60: $\frac{1}{2}$ life is 5.3 years (Used in medical radiology)

How long does it take for 90% to decay?? $\equiv N(t) = \frac{1}{10} N_0$

$kt_{1/2} = \ln 2$ so $k = \frac{\ln 2}{5.3}$

$N(t) = N_0 e^{-\frac{\ln 2}{5.3} t} = \frac{1}{10} N_0$ so $e^{-\frac{\ln 2}{5.3} t} = \frac{1}{10}$

Take \ln : $-\frac{\ln 2}{5.3} t = \ln \frac{1}{10} = -\ln 10$

$t = \frac{5.3 \ln 10}{\ln 2} \approx 17.606$ years.