

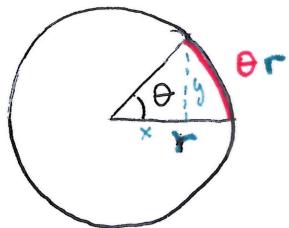
Lecture XXXII: § 9.6 Review of Trigonometry

§ 9.2 The derivatives of sine & cosine

§ 9.3 Integrals of $\sin(x)$, $\cos(x)$

§ 9.4 The other 4 trigonometric functions.

§1 Basics:



$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{for any } 0 \leq \theta \leq 2\pi$$

θ measured in radians so Full circle $0 \leq \theta \leq 2\pi$

• Periodicity: $\cos(\theta + 2\pi) = \cos \theta$

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\tan = \frac{\sin \theta}{\cos \theta}, \quad \cot(\theta) = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc(\theta) = \frac{1}{\sin \theta}$$

Parity: $\sin(-\theta) = -\sin \theta$
 $\cos(-\theta) = \cos \theta$
 $\tan(-\theta) = -\tan(\theta)$

Area: $\pi r^2 \frac{\theta}{2\pi} = \frac{\theta r^2}{2}$

Length: $\theta r = 2\pi r \frac{\theta}{2\pi}$

§2 Trig Identities:

① $\sin^2 \theta + \cos^2 \theta = 1$

$$\left\{ \begin{array}{l} \tan^2 \theta + 1 = \sec^2 \theta \quad (1/\cos^2 \theta) \\ 1 + \cot^2 \theta = \csc^2 \theta \quad (1/\sin^2 \theta) \end{array} \right.$$

$$\left\{ \begin{array}{l} \tan^2 \theta + 1 = \sec^2 \theta \quad (1/\cos^2 \theta) \\ 1 + \cot^2 \theta = \csc^2 \theta \quad (1/\sin^2 \theta) \end{array} \right.$$

② Addition formulas:

$$\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

$$\cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$$

$$\text{so } \tan(\theta + \varphi) = \frac{\sin \theta \cos \varphi + \cos \theta \sin \varphi}{\cos \theta \cos \varphi - \sin \theta \sin \varphi} = \frac{\tan \theta + \tan \varphi}{1 - \tan \theta \tan \varphi}$$

divide by $\cos \theta \cos \varphi$
both num & denom

• Alt rule: $e^{i\theta} := \cos \theta + i \sin \theta \quad (i^2 = -1) \quad [\text{Complex Numbers}]$

$$\text{so } e^{i(\theta + \varphi)} = \cos(\theta + \varphi) + i \sin(\theta + \varphi)$$

$$\begin{aligned} e^{i\theta} e^{i\varphi} &= (\cos \theta + i \sin \theta) \cdot (\cos(\varphi) + i \sin(\varphi)) \\ &= \cos \theta \cos \varphi + i \cos \theta \sin \varphi + i \sin \theta \cos \varphi + i^2 \sin \theta \sin \varphi \end{aligned}$$

$$= \underline{\omega s\theta \cos\varphi - \sin\theta \sin\varphi} + i(\underline{\omega s\theta \sin\varphi + \sin\theta \cos\varphi})$$

$$= \cos(\theta + \varphi) + i \sin(\theta + \varphi)$$

Special case:

(1) Double angle: $\sin 2\theta = 2 \sin \theta \cos \theta$

$$\cos 2\theta = \omega^2 \theta - \sin^2 \theta = \omega^2 \theta - (1 - \cos^2 \theta) = 2\cos^2 \theta - 1$$

(2) Half-angle:

$$2\omega^2 \theta = 1 + \cos 2\theta.$$

$$2 \sin^2 \theta = 1 - \cos 2\theta.$$

$$\Rightarrow \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$$

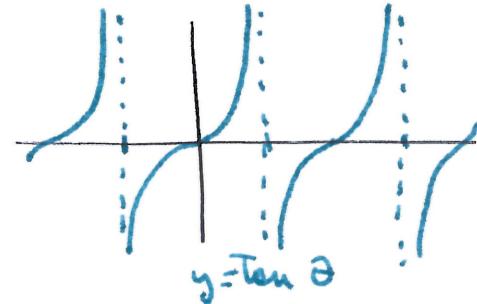
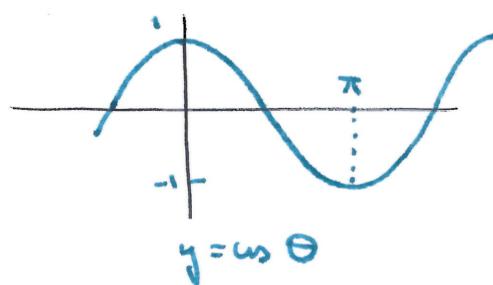
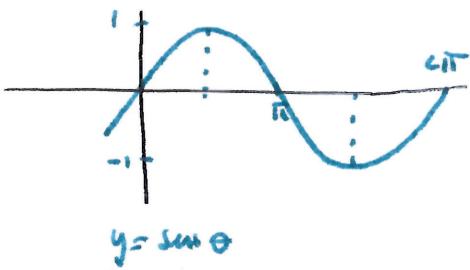
$$= (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta$$

$$(Use \sin^2 \theta + \cos^2 \theta = 1 \& \cos^2 \theta - \sin^2 \theta = \tan^2 \theta)$$

useful for integration!

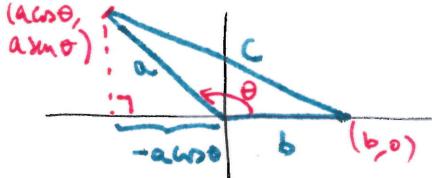
$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}.$$

§ 3 Graphs:



not defined for $\theta = \frac{(2k+1)\pi}{2}$

§ 4 Law of cosines



Law of cosines: $c^2 = a^2 + b^2 - 2ab \cos \theta$

$$\text{for } \frac{\pi}{2} \leq \theta < \pi$$

Proof:

$$\begin{aligned} c^2 &= (a \sin \theta)^2 + (b - a \cos \theta)^2 \\ &= a^2 \sin^2 \theta + b^2 + a^2 \cos^2 \theta - 2ab \cos \theta \\ &= a^2 (\underbrace{\sin^2 \theta + \cos^2 \theta}_{=1}) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta \end{aligned}$$

Note: For $\theta = \frac{\pi}{2}$, we use the usual formula since $\cos \frac{\pi}{2} = 0$.

§ 5 Derivatives & Integrals

• $\frac{d}{dx} \sin(x) = \cos(x)$	$\frac{d}{dx} \tan(x) = \sec^2(x)$	$\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$
• $\frac{d}{dx} \cos(x) = -\sin(x)$	$\frac{d}{dx} \cot(x) = -\csc^2(x)$	$\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$

3f/ Quotient rule / Chain Rule .

$$\begin{aligned} \cdot \int \csc(x) dx &= -\cot(x) + C & \int \sec^2(x) dx &= \tan(x) + C \\ \cdot \int \sec(x) dx &= \ln|\sec(x)| + C & \int \csc^2(x) dx &= -\cot(x) + C \\ \cdot \int \sec(x) \tan(x) dx &= \sec(x) + C & \int \csc(x) \cot(x) dx &= -\csc(x) + C \end{aligned}$$

Extra formulas:

$$(1) \int \tan(x) dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|u| + C = -\ln(\cos(x)) + C$$

$$(2) \int \cot(x) dx = \int \frac{\cos(x)}{\sin x} dx = \int \frac{du}{u} = \ln|\sin(x)| + C$$

$$(3) \int \sec(x) dx = \ln(\sec(x) + \tan(x)) + C$$

$$\text{Why? } \int \sec(x) dx = \int \frac{\sec(x)}{\sec(x) + \tan(x)} \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} dx = \int \frac{\sec^2 x + \sec(x)\tan(x)}{\sec(x) + \tan(x)} dx$$

$$= \int \frac{du}{u} = \ln(\sec(x) + \tan(x)) + C.$$

$$u = \sec(x) + \tan(x)$$

$$du = (\sec(x)\tan(x) + \sec^2(x))dx$$

$$(4) \int \csc(x) dx = -\ln(\csc(x) + \cot(x)) + C$$

$$\text{Why? } \int \csc(x) \frac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)} dx = \int \frac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot(x)} dx$$

$$= -\int \frac{du}{u} = -\ln(\csc(x) + \cot(x)) + C.$$

$$u = \csc(x) + \cot(x)$$

$$du = (-\csc(x)\cot(x) - \csc^2(x))dx$$

$$(5) \int \cos^2 \theta d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$\int \sin^2 \theta d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C.$$