

Lecture XXXIII §9.5 The inverse trigonometric functions

GOAL: Prove the following integration formulas

$$(1) \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C \quad \& \quad (2) \int \frac{dx}{1+x^2} = \tan^{-1}x + C$$

Along the way, define inverse trig. functions

Idea: substitution $x = \sin(u)$ for (1) & $x = \tan u$ for (2)
 $\therefore dx = \cos(u) du$ $dx = \sec^2 u du$

$$(1) \sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \sqrt{\cos^2 u} = \cos u$$

$$\int \frac{\cos u \, du}{\cos u} = \int du = u + C = \underset{\substack{\uparrow \\ \text{to be definite}}}{\sin^{-1}(x)} + C$$

$$(2) 1+x^2 = 1+\tan^2 u = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = (\cos^2 u)^{-1} = \sec^2 u$$

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2 u \, du}{\sec^2 u} = \int du = u + C = \tan^{-1}(x) + C \quad \underset{\substack{\uparrow \\ \text{to be defined.}}}{}$$

\Rightarrow We are forced to invert $\sin(x)$ & $\tan(x)$!

§1 Inverse sine = \sin^{-1} or \arcsin^{-1} (2 notations)

Def.: $x = \sin(u)$ if and only if $u = \arcsin(x)$, or $u = \sin^{-1}(x)$

That is $x = \sin(\sin^{-1}u)$ & $u = \sin^{-1}(\sin u)$

These are inverse functions with respect to composition

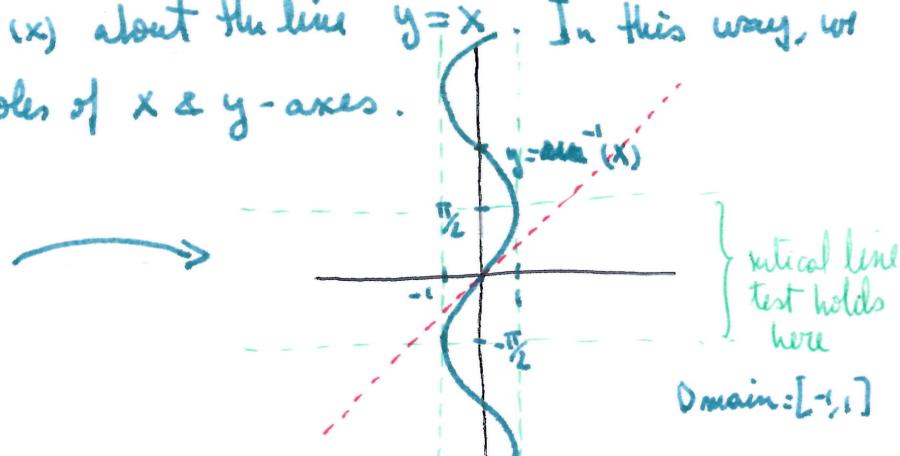
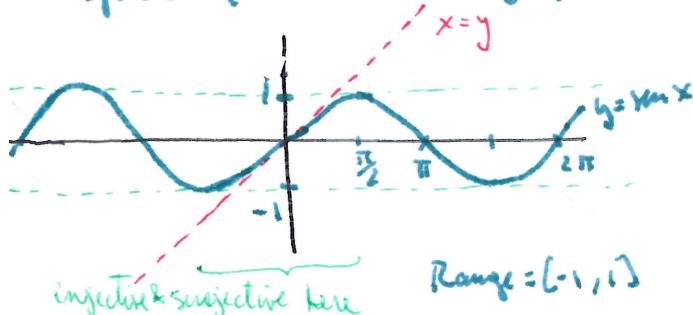
Domain of \sin^{-1} = Range of \sin = $[-1, 1]$

\sin is periodic, so to define \sin^{-1} we restrict to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ = ^{an} interval where $\sin(x)$ is injective & surjective

Q: Why? A: Need vertical line test to hold!

Q: How to graph $y = \sin^{-1} x$?

A: Reflect the graph of $y = \sin(x)$ about the line $y = x$. In this way, we reflect (or interchange) the roles of x & y -axes.



Include: $\sin^{-1}: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

Remark: $y = \sin(x)$ is smooth & has tangents, so will $y = \sin^{-1}(x)$ (we just reflected the curve!) Thus, $\sin^{-1}(x)$ is differentiable as a function

- We compute $\frac{d}{dx} \sin^{-1}(x)$ with implicit differentiation!

$$y = \sin^{-1}(x) \quad \text{so} \quad \sin(y) = x \quad \Rightarrow \cos(y) \cdot y' = 1 \\ \frac{d}{dx} \quad \quad \quad y' = \frac{1}{\cos y}.$$

But if $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$: $\cos(y) \geq 0$ so $\cos(y) = \sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

We conclude: $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$. & $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$

Note: For this formula to make sense, we are forced to have $-1 \leq x \leq 1$, which is true since Domain $\sin^{-1} = [-1, 1]$ $\Rightarrow x \leq 1$ don't consider derivatives at $x = \pm 1$.

3.2 Inverse Tangent = $\tan^{-1}(x)$

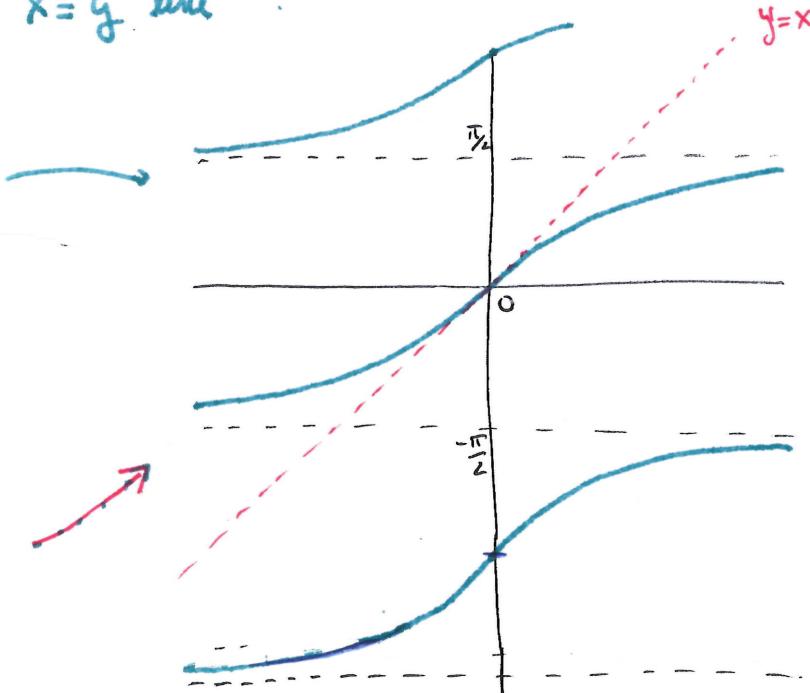
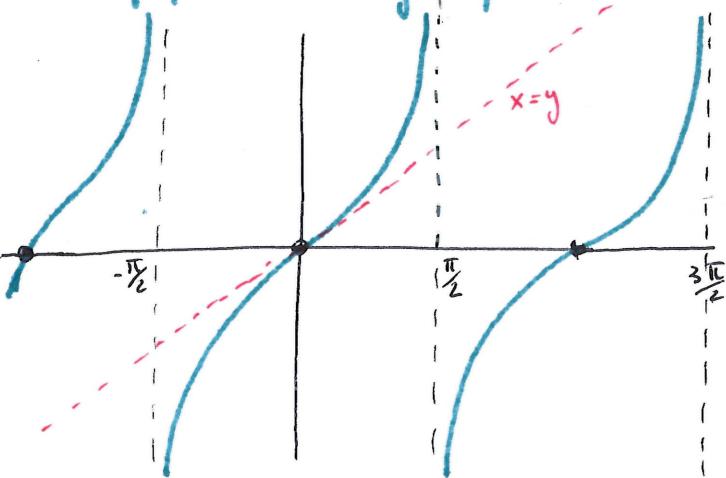
Def: $x = \tan(u)$ if and only if $u = \arctan(x)$, or $u = \tan^{-1}(x)$

That is $x = \tan(\tan^{-1}(x))$ & $u = \tan^{-1}(\tan(u))$.

These are inverse functions with respect to composition (as \sin & \sin^{-1} were)

Domain Tan = $(-\frac{\pi}{2}, \frac{\pi}{2})$, Range Tan = \mathbb{R} , $\tan(x)$ is periodic with period $= \pi$.

We graph \tan^{-1} by reflection about $x=y$ line.



• We need to pick a branch to define $\tan^{-1}(x)$

$$\tan^{-1}: \mathbb{R} \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Note: • $\tan(x)$ is increasing, so is $\tan^{-1}(x)$
 • $\tan^{-1}(0) = 0$
 • $\tan^{-1}(x)$ has well defined tangents Y, so $\tan^{-1}(x)$ is a differentiable function

• We compute $\frac{d}{dx} \tan^{-1}(x)$ with implicit differentiation:

$$y = \tan^{-1}(x) \quad \text{so} \quad x = \tan(y) \quad \Rightarrow \quad 1 = \sec^2(x) y'$$

$$\text{So } y' = \frac{1}{\sec^2(y)} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\sec^2 y = \frac{1}{\cos^2 y} = 1 + \frac{\sin^2 y}{\cos^2 y} = 1 + \tan^2 y$$

(Conclusion:

$$\boxed{\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}}$$

and

$$\boxed{\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C}$$

Exercises: (1) $\frac{d}{dx} \sin^{-1}\left(\frac{x}{5}\right) = \frac{1}{\sqrt{1-(\frac{x}{5})^2}} \cdot \frac{1}{5} = \frac{1}{\sqrt{25-x^2}}$

(2) $\frac{d}{dx} \sin^{-1}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{1-(\frac{1}{x})^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{-1}{x\sqrt{x^2-1}}$

(3) $\frac{d}{dx} \tan^{-1}\left(\sqrt{1+x^2}\right) = \frac{1}{1+(1+x^2)} \cdot \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}(2+x^2)}$

(4) $\frac{d}{dx} \left(\tan^{-1}(x) + \ln \sqrt{1+x^2} \right) = \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}} \cdot \frac{x}{\sqrt{1+x^2}} = \frac{1+x}{(1+x^2)^{3/2}}$

(5) $\int \frac{dx}{\sqrt{1-16x^2}} = \int \frac{du}{4\sqrt{1-u^2}} = \frac{1}{4} \sin^{-1}(4x) + C$

(6) $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}(x) \Big|_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3}) - \tan^{-1}(0) = \boxed{\frac{\pi}{3}}$

(7) $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{-du}{u\sqrt{u^2-1}} = \int \frac{-du}{u\sqrt{\frac{1-u^2}{u^2}}} = \int \frac{-du}{\sqrt{1-u^2}} = \sin^{-1}\left(\frac{1}{x}\right) + C$

$u = \frac{1}{x} \Rightarrow x^2 = u^{-2}$
 $du = -\frac{1}{x^2} dx \Rightarrow \frac{dx}{x} = -x du = -\frac{du}{u}$

(8) $\int \frac{15x^4 dx}{\sqrt{1-x^{10}}} = \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3} \sin^{-1}(x^5) + C.$

$u = x^5$
 $du = 5x^4 dx$