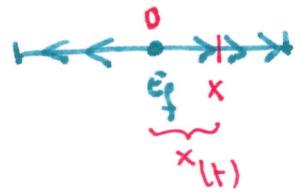


TODAY: Model the motions of vibrations that produce sounds  
• oscillations or waves of periodic motions.

### §1 Simple Harmonic Motions

Def: If an object or point moves back & forth in a straight line (say the  $x$ -axis) so that the force required to move it back to equilibrium ( $x=0$ ) is proportional to the distance from equilibrium, we say we have a simple harmonic motion.



$$F(x) = m \frac{d^2x}{dt^2} = -kx(t) \quad \text{for } k > 0$$

$$\text{gives } \frac{d^2x}{dt^2} = -\frac{k}{m} x(t)$$

Customary: write  $c = a^2 > 0$  (to emphasize it's positive!)

Equation for simple harmonic motion: (\*)  $\frac{d^2x}{dt^2} + a^2 x(t) = 0 \quad \text{for } a > 0$

$$\begin{cases} x(0) = x_0 & (\text{initial position}) \\ x'(0) = v_0 & (\text{initial velocity}) \text{ (typically } x_0=0) \end{cases}$$

Q: What do the solutions look like?

Eg:  $a=1 \quad x'' + x = 0 \quad \Rightarrow \text{Sols: } x = x_0 \sin(t), \quad x = v_0 \cos(t)$

In general:  $x = \alpha \sin(t) + \beta \cos(t) \quad \text{for any } \alpha, \beta \in \mathbb{R}$ .

Prop: The solutions to (\*) (SHM eqn) are of the form

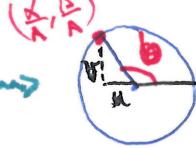
$$x(t) = A \sin(at + b) \quad \text{for some } A, b \in \mathbb{R}. \quad \Rightarrow 2 \text{ parameters}$$

Proof:  $\sin(at)$  &  $\cos(at)$  satisfy  $x'' + a^2 x = 0$ . Furthermore (Prob 19 of Additional Problems Ch 9), the general solution has the form

$$x(t) = \alpha \sin(at) + \beta \cos(at). \quad \Rightarrow 2 \text{ parameters } (\alpha, \beta)$$

Since we don't want a trivial solution ( $x \equiv 0$ ), we can assume  $\alpha \neq \beta$  are not both 0. Take  $A = \sqrt{\alpha^2 + \beta^2} \neq 0$

The point  $(\frac{\alpha}{A}, \frac{\beta}{A})$  lies in the unit circle and



$$\begin{cases} \text{For some } b \in [0, 2\pi) \\ u = \frac{\alpha}{A} = \cos \theta \\ v = \frac{\beta}{A} = \sin \theta \end{cases}$$

$$\text{Now } x(t) = \alpha \sin(\omega t) + \beta \cos(\omega t) = A \cos(b) \sin(\omega t) + A \sin(b) \cos(\omega t)$$

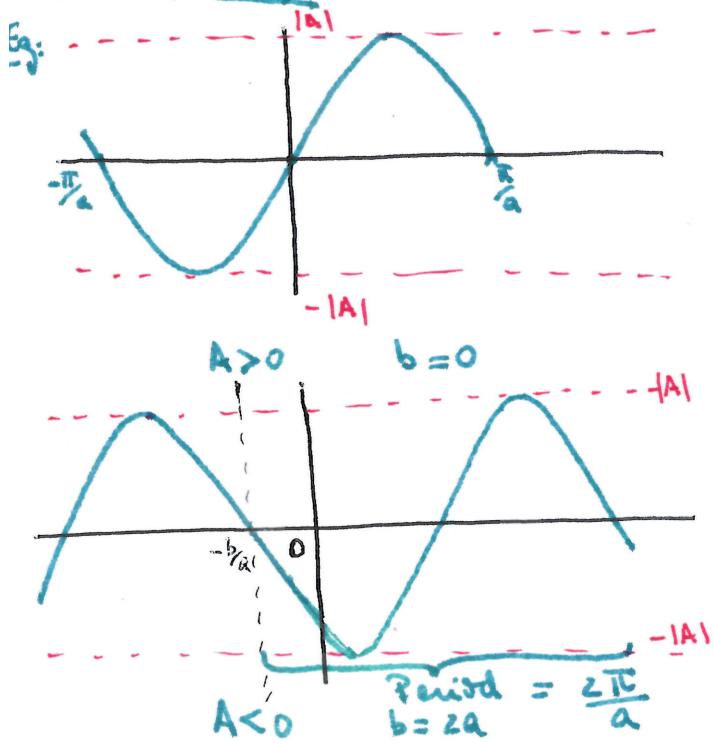
$$= A (\cos(b) \sin(\omega t) + \sin(b) \cos(\omega t)) \stackrel{\text{formula for } \sin(a+b)}{=} A \sin(\omega t + b)$$

Conclusion: general solution is as we claimed (A need not be positive!)

Remark: Can check this is a solution for LSO as well.

$$\frac{d}{dt} A \sin(\omega t + b) = a A \cos(\omega t + b) \Rightarrow \frac{d^2 x(t)}{dt^2} = -a^2 A \sin(\omega t + b) = -a^2 x(t)$$

Graph of  $x(t)$ :



- Oscillates between  $|A|$  &  $-|A|$
- Periodic:  $a(t+\theta)+b = at+b+a\theta = at+b+2\pi$   
so period is  $\frac{2\pi}{\omega}$
- In general:  $x(t) = A \sin(\omega t + b) = A \sin(\omega t + \frac{b}{\omega})$

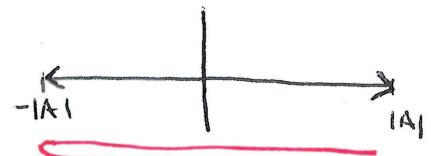
⇒ shift the graph by  $-\frac{b}{\omega}$  units  
• rotate about x-axis if  $A < 0$ .

Note: Period = time it takes to get back to the same position. ( $= \frac{2\pi}{\omega}$  sec)

Names: .  $|A|$  = amplitude

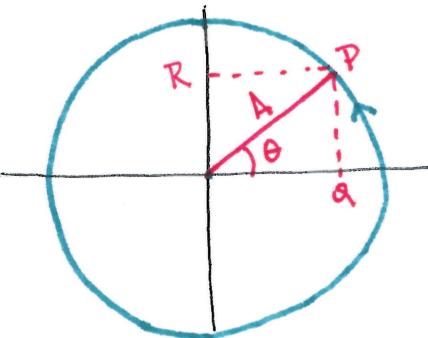
.  $T = \frac{2\pi}{\omega}$  = period (= time of a cycle)

.  $f = \frac{1}{T} = \frac{\omega}{2\pi}$  = frequency (= cycles per sec)



time to travel this cycle  $= \frac{2\pi}{\omega}$

§ 3 Another perspective on periodic motion Assume  $A > 0$



The particle P moves around the circle of radius A with constant angular velocity  $\frac{d\theta}{dt} = \omega$  rad/sec

as soln:  $\theta = at + b$  (by integration) for some  $b$

Write:  $Q = \text{projection to } x\text{-axis} = A \cos \theta$   
 $R = \text{projection to } y\text{-axis} = A \sin \theta$

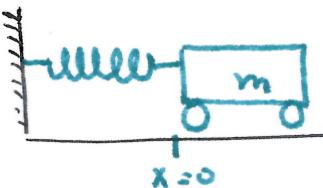
So the equations of motion of Q & R are

$$x = A \cos(at + b) \\ = A \sin(at + b + \frac{\pi}{2})$$

$$y = A \sin(at + b)$$

### § Examples:

#### I SHM for a spring:



Push away  
from the wall

Equilibrium

Assume: friction & air resistance are negligible

$$\begin{matrix} F = m \frac{dv}{dt} = m \frac{d^2x}{dt^2} = -kx \\ \downarrow \\ \text{Newton's Law} \end{matrix}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \\ = a^2 > 0$$

Solution:  $x(t) = \alpha \sin(at) + \beta \cos(at)$   $\Rightarrow a = \sqrt{\frac{k}{m}} > 0$

Initial cond:  $\begin{cases} x(0) = x_0 \\ \frac{dx}{dt}(0) = 0 \end{cases}$

$$x'(t) = a\alpha \cos(at) - a\beta \sin(at) \quad \text{so} \quad x'(0) = a\alpha = 0 \quad \text{gives } \alpha = 0$$

$$x(0) = \beta = x_0$$

Conclusion:  $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}} t\right)$

$$\text{Or } \sin(2\pi \frac{t}{T}) = \cos \theta \text{ and } x(t) = x_0 \sin\left(\sqrt{\frac{k}{m}} t + \frac{\pi}{2}\right)$$

Amplitude  $\approx x_0 > 0$  initial position

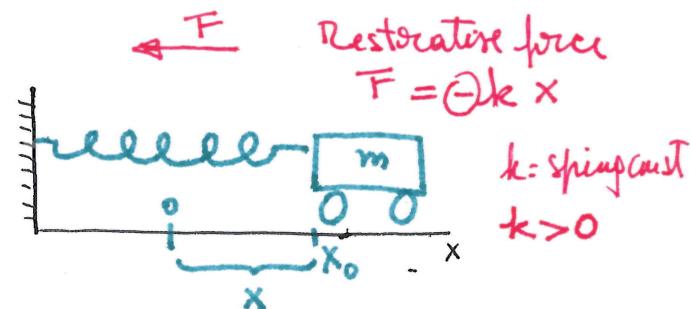
$$\cdot \text{Period: } T = \frac{2\pi}{a} = 2\pi \sqrt{\frac{m}{k}}$$

$$\cdot \text{Frequency: } f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Note. If stiffness  $k$  of the spring increases, then the frequency of the vibration increases.

If the mass  $m$  of cart increases, the frequency decreases.

(Computations agree with expectation)

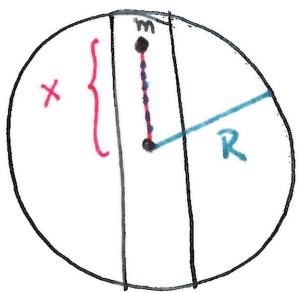


Let go of the cart with initial velocity  $v_0 = 0$

[ $-v_0$  because  $x$  is positive against the direction of the force]

## ⑤ Holes on the earth:

- We bore a tunnel straight through the center of the earth from one side to the other
- We drop a body of mass  $m$  into the tunnel
- Earth has uniform density & its a sphere of radius  $R = 4000 \text{ mi}$
- GOAL: Motion of the body induced by the gravity?



$$\downarrow F = -m \frac{d^2x}{dt^2}$$

• Force of gravity acts as if all the earth is concentrated at the center of mass (=center because uniform density)

Model:  $F = -kx \Rightarrow x'' + \frac{k}{m}x = 0$

At the surface  $F = -mg \Rightarrow$  acceleration of gravity  $\rightarrow x = R$

$$\text{So } -mg = -kR \Rightarrow k = \frac{mg}{R} \Rightarrow \frac{k}{m} = \frac{g}{R}$$

Eqn:  $x'' + \frac{g}{R}x = 0$

is "independent" of the mass  $m$ .

Conclusion: If we drop the object from the surface through the hole, it will exhibit SHM (disregarding the air friction, etc.) with

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{R}}} = 2\pi \sqrt{\frac{R}{g}} \approx 89 \text{ min}$$

$$\begin{cases} R \approx 4000 \text{ mi} \\ g = 32 \text{ ft/sec}^2 \\ 1 \text{ mi} = 5280 \text{ ft.} \end{cases}$$

• Time to get to center =  $\frac{89}{4} \approx 22 \text{ min.}$

• Amplitude = ? Claim =  $|A| = \text{Radius of earth.}$

$$x(t) = A \sin(\omega t + b)$$

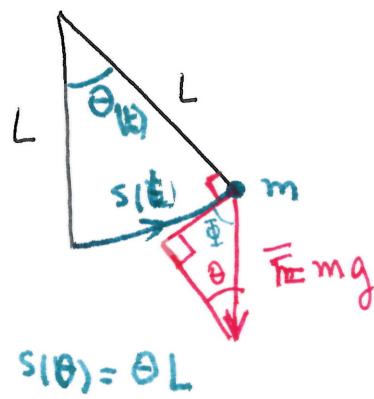
$$x(0) = A \sin b = R$$

$$x'(t) = A\omega \cos(\omega t + b) \Rightarrow x'(0) = A\omega \cos b = 0 \Rightarrow \cos b = 0$$

If  $\cos b = 0$  then  $\sin b = \pm 1$ .  $\Rightarrow |A| = \left| \frac{R}{\sin b} \right| = |R| = R$ .

(Already saw this in Example ① (SHM of spring))

III

Pendulum:

- Bob (weight) suspended at the end of a light string of length L.

- Allowed to swing back and forth under the action of gravity

$F = mg$ , but not in the direction of the movement.

$F$  has 2 components: . i tangential to the movement =  $F_{tan}$

• ii perpendicular  $\longrightarrow$   $= F_{perp}$

$$F_{tan} = -F \sin \theta = -mg \sin \theta.$$

Equation,  $\frac{d^2 s}{dt^2} = \frac{d^2}{dt^2}(\theta L) = \left(\frac{d^2 \theta}{dt^2}\right) L$

Newton's Law  $F_{tan} = m \frac{d^2 s}{dt^2} = mL \theta''$  gives

$$mL \theta'' = -mg \sin \theta \Rightarrow \boxed{\theta'' + \frac{g}{L} \sin \theta = 0}$$

For small values of  $\theta$ ,  $\sin \theta \approx \theta$  (since  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ ) & we get an eqn of SHM:  $\theta'' + \frac{g}{L} \theta = 0$  with period  $= \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi \sqrt{\frac{L}{g}}$

This gives an approximation!

Reality: Period of oscillation depends on the amplitude of motion. This is the source of the circular error in pendulum clocks.

§6 Alternative proof of Prop:

Input  $\frac{d^2 x}{dt^2} + a^2 x = 0$

Output: general solution  $x(t) = A \sin(at+b)$

Write  $\frac{d^2 x}{dt^2} = \frac{d v}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$  by Chain Rule

$$\text{Eqn becomes } v \frac{dr}{dx} + a^2 x = 0 \quad \Rightarrow \quad v dr = -a^2 x dx$$

(separation of variables!)

We integrate to get:

$$\frac{v^2}{2} = -\frac{a^2}{2} x^2 + C' \quad \text{so} \quad r^2 + a^2 x^2 = 2C' =: C \text{ constant.}$$

$$\text{If } v(x_0) = 0, \text{ we get } v^2(x_0) + a^2 x_0^2 = C \quad \text{so} \quad C = a^2 x_0^2$$

$$\text{Set } A = x_0 > 0 \text{ to get } v^2(x) + a^2 x^2 = a^2 A^2 \quad \text{or} \quad v^2(x) = a^2 (A^2 - x^2)$$

$$\Rightarrow \frac{dx}{dt} = v = \pm a \sqrt{A^2 - x^2} \quad \text{or} \quad \frac{dx}{\sqrt{A^2 - x^2}} = \pm a dt \quad (\text{sign depends on velocity})$$

Assume  $r > 0$ , & integrate:

$$(\text{LHS}): \int \frac{dx}{\sqrt{A^2 - x^2}} = \frac{1}{A} \int \frac{dx}{\sqrt{1 - \left(\frac{x}{A}\right)^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1}(u) = \sin^{-1}\left(\frac{x}{A}\right)$$

$$(\text{RHS}): \int a dt = at + b$$

$$\text{We get } \sin^{-1}\left(\frac{x}{A}\right) = at + b \quad \text{so} \quad x = A \sin(at + b). \quad \square$$

3.7 General solution to  $y'' + a^2 y = 0$ . (\*) is  $y(t) = C_1 \cos at + C_2 \sin at$ .

(1) If  $f(t), g(t)$  are solutions, and  $\alpha, \beta$  are real numbers, then

$$y = \alpha f(t) + \beta g(t) \text{ is also a solution.}$$

$$\text{Pf/ Taking derivative is linear: } \begin{aligned} y'' &= \alpha f'' + \beta g'' \\ a^2 y &= \alpha^2 f + \beta^2 g \end{aligned}$$

$$y'' + a^2 y = \alpha(f'' + a^2 f) + \beta(g'' + a^2 g) = \alpha \cdot 0 + \beta \cdot 0 = 0 \quad \checkmark$$

(2) If  $y = f(t)$  is a solution, then  $a^2 f(t)^2 + (f'(t))^2 = \text{constant}$ .

$$\text{Pf/ Take derivative: } \begin{aligned} (a^2 f(t)^2 + (f'(t))^2)' &= a^2 2 f(t) f' + 2 f'(t) f''(t) \\ &= 2 f'(t) (a^2 f(t) + f''(t)) = 2 f'(t) \cdot 0 = 0. \quad \checkmark \end{aligned}$$

(3) If  $y(t)$  is a solution &  $y(0) = y'(0) = 0$ , then  $y(t) = 0$  for all  $t$ .

$$\text{Pf/ By (2)} \quad a^2 f(t)^2 + (f'(t))^2 = \text{const} = a^2 f(0)^2 + (f'(0))^2 = 0.$$

$$\text{So } \underbrace{a^2 f(t_0)^2}_{\geq 0} + \underbrace{(f'(t_0))^2}_{\geq 0} = 0 \quad \text{only option is } f(t_0) = f'(t_0) = 0. \text{ for all } t.$$

$$(1) \quad f(t) = y(t) - \underbrace{\frac{1}{a} y'(0)}_{\text{in IR}} \sin(at) - \underbrace{y(0)}_{\text{in IR}} \cos(at).$$

By (1).  $y(t)$  is a solution (lin. comb of solns)

$$\text{so } f'(t) = y'(t) - \frac{1}{a} y'(0) \cos(at) + a y(0) \sin(at)$$

$$\text{so } f'(0) = y'(0) - \frac{1}{a} y'(0) + 0 = 0$$

$$f(0) = y(0) - 0 - y(0) = 0$$

$$\text{By (3) } f(t) = 0 \text{ for all } t \quad \text{so } y(t) = \boxed{\frac{1}{a} y'(0) \sin(at)}_{\text{const}} + \boxed{y(0) \cos(at)}_{\text{const}}.$$

So any solution has the general form we want!