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Lecture XXXV: §10.1 The basic formulas
 §10.2 The method of substitution
 §10.3 Certain trigonometric integrals

§1 The Basic formulas

Def.: An elementary function is one built from x^a , e^x , $\ln(x)$, $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$ & $\tan^{-1}(x)$.

Example: $\tan^{-1}\left(\frac{\ln(x^2 + \cos^2(x))}{e^x + \sin\sqrt{x^{2.5}+1}}\right)$

Note: • Simple rules for differentiating building blocks + Chain/Prod Rule gives easy method to differentiate elementary functions

- Integration is more subtle: no systematic method to integrate the elem. functions & answer need not be an elem. function. We have a recognition problem (what method to apply? how to apply it?)

Eg: $\text{Li}(x) = \int_2^x \frac{dt}{\ln t}$; $\int_0^x e^{-t^2} dt$ are not expressible as elem. function

[Appendix A9]

- 15 basic formulas (see Handout, book page 335 & 336)
(last page of notes)

§2 Method of substitution

Recall: Substitution is the analogue of the Chain Rule for integration

⇒ Recognition Problem!

$$\text{Substitution: } \int_a^b f'(g(x)) g'(x) dx \stackrel{u=g(x)}{=} \int_{g(a)}^{g(b)} f'(u) du = f(u) \Big|_{g(a)}^{g(b)} = f(g(b)) - f(g(a))$$

$du = g'(x) dx$

Example ① $\int x e^{-x^2} dx = \int e^u \frac{du}{-2} = -\frac{e^u}{2} + C = -\frac{e^{-x^2}}{2} + C$

$$\begin{aligned} u &= -x^2 \\ du &= -2x dx \end{aligned}$$

$$\begin{aligned} &= \int u^{-1/2} du = 2u^{1/2} + C = 2(1+\sin(x))^{1/2} + C \end{aligned}$$

$$\begin{aligned} u &= 1+\sin(x) \\ du &= \cos(x) dx \end{aligned}$$

Note: $u = \cos(x)$ $\Rightarrow u = \sqrt{1+\sin(x)}$ won't work!

$$\textcircled{3} \quad \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C$$

$u = \ln x$
 $du = \frac{1}{x} dx$

$$\textcircled{4} \quad \int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{3 \sqrt{1-\frac{4}{9}x^2}} = \int \frac{1}{2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1}(u) + C$$

\downarrow
 $u = \frac{2}{3}x$
 $\left(\frac{2}{3}x\right)^2 du = \frac{4}{3} dx$

$$= \frac{1}{2} \sin^{-1}\left(\frac{2}{3}x\right) + C$$

$$\textcircled{5} \quad \int \frac{x dx}{\sqrt{9-4x^2}} = \int u^{\frac{1}{2}} \frac{du}{(-8)} = \frac{1}{-8} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{\sqrt{9-4x^2}}{4} + C$$

$u = 9-4x^2$
 $du = -8x dx$

§ 10.3 Certain Trigonometric functions:

GOAL: Find a method to integrate :

$$\textcircled{1} \quad \int \sin^m(x) \cos^n(x) dx$$

$$\textcircled{2} \quad \int \tan^m(x) \sec^n(x) dx \quad \text{for } m, n \geq 0 \text{ integers}$$

$$\textcircled{3} \quad \int \cot^m(x) \csc^n(x) dx$$

• Answer will depend on the parity of m & n .

Examples: ① $\int \sin^m(x) \cos(x) dx = \frac{\sin^{m+1}(x)}{m+1} + C \quad (n=1)$

$$\textcircled{2} \quad \int \tan^m(x) \sec^2(x) dx = \frac{\tan^{m+1}(x)}{m+1} + C \quad (n=2)$$

$$\textcircled{3} \quad \int \cot^m(x) \csc^2(x) dx = + \frac{\cot^{m+1}(x)}{m+1} + C \quad (n=2)$$

Q: What about other values of n ? Let's start with ①, the others are similar

CASE A m or n are odd (eg 1, 3, 5, ...)

We use trigonometric identities to turn the integrand into one of the following : $\cos^a(x) \sin(b)(x) \quad \text{or} \quad \sin^a(x) \cos(b)(x)$.

Example: for n odd write $n = 2k+1$ for some $k \geq 0$ integer.

then $\cos^{2k+1}(x) = (\cos^2(x))^k \cos(x) = (1 - \sin^2(x))^k \cos(x)$

exp. rules $\cos^2(x) = 1 - \sin^2(x)$

Use Binomial Theorem to write $(1 - \sin^2(x))^k = 1 - k \sin^2(x) + \binom{k}{2} \sin^4(x) - \dots + (-1)^k \sin^{2k}(x)$

so $\sin^m(x) \cdot \cos^n(x) dx = \sum_{j=0}^k (-1)^j \underbrace{\sin^{2j}(x) \cos x}_{\hookrightarrow \text{we know how to integrate this!}} dx$

The construction for m odd is similar.

Ex: (i) $\int \sin^3(x) dx = \int \sin(x) \sin^2(x) dx = \int \sin(x) (1 - \cos^2(x)) dx$
 $= \int \sin(x) dx - \int \sin(x) \cos^2(x) dx = -\cos(x) + \frac{\cos^3(x)}{3} + C.$

Ex: (ii) $\int \sin^2(x) \cos^5(x) dx = \int \sin^2(x) \cos(x) (\cos^2(x))^4 dx$
 $= \int \sin^2(x) \cos(x) \underbrace{(1 - \sin^2(x))^4}_{= 1 - 2 \sin^2(x) + \sin^4(x)} dx = \int \sin^2(x) \cos(x) dx$
 $- 2 \int \sin^4(x) \cos(x) dx + \int \sin^6(x) \cos(x) dx = \frac{\sin^3(x)}{3} - \frac{2 \sin^5(x)}{5} + \frac{\sin^7(x)}{7} + C$

CASE B Both m & n are even. ($m = 2k$, $n = 2l$ for $k, l \geq 0$ integers)

Use half-angle formulas!

$$\begin{cases} \cos^2(x) + \sin^2(x) = 1 \\ \cos^2(x) - \sin^2(x) = \cos(2x) \end{cases}$$

so
$$\begin{cases} 2\cos^2(x) = 1 + \cos(2x) & \text{(add)} \\ 2\sin^2(x) = 1 - \cos(2x) & \text{(subtract)} \end{cases}$$

Example: $\int \cos^4(x) dx = \int (\cos^2(x))^2 dx = \int (1 + \cos(2x))^2 dx$
 $= \frac{1}{4} \int 1 + 2\cos(2x) + \cos^2(2x) dx = \frac{1}{4} \left(x + \sin 2x + \int \cos^2(2x) dx \right)$

$$\int \cos^2(u) du = \frac{1}{2} \int 1 + \cos(2u) du = \frac{u}{2} + \frac{\sin 2u}{4} + C$$

so $\int \cos^4(x) dx = \frac{1}{4} x + \frac{\sin 2x}{4} + \frac{1}{4} \int \cos^2(u) \frac{du}{2} \stackrel{u=2x}{=} \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8} \left(\frac{2x}{2} + \frac{\sin 4x}{4} \right) + C = \frac{3}{8} x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

As before $\cos^{2k}(x) \sin^{2l}(x) = (\cos^2(x))^k (\sin^2(x))^l$

& use $\frac{1}{2}$ angle formulas + Binomial Thm.

② Again : 3 cases + Trig identities . $1 + \tan^2(x) = \sec^2(x)$. (n even) (4)

- A. m odd
- B. n even
- C. neither

$$\tan^3 x = \tan^2 x \cdot \tan x$$

$m=3$ odd

$$\begin{aligned} \tan^2(x) &= \sec^2(x) - 1 \quad (\text{m odd}) \\ d(\tan(x)) &= \sec^2(x) dx \\ d(\sec(x)) &= \sec(x) \tan(x) dx. \end{aligned}$$

A. $\int \tan^3(x) \sec(x) dx \stackrel{\uparrow}{=} \int (\sec^2(x) - 1) \tan(x) \sec(x) dx$ keep $\tan(x)$

$$= \int \underbrace{\tan(x) \sec(x)}_{d(\sec(x))} \sec^2(x) dx - \int \tan(x) \sec(x) dx = \frac{\sec^3(x) - \sec(x)}{3} + C$$

B. $\int \tan^4(x) \sec^6(x) dx \stackrel{\uparrow}{=} \int \tan^4(x) (1 + \tan^2(x))^2 \sec^2(x) dx$ 6 = 2+2+2 keep one $\sec^2(x)$.

$$\begin{aligned} &= \int \tan^4 x \cdot \sec^2 x | \cdot \tan^4(x) + 2 \cdot \tan^2(x) + 1 dx \quad \text{Binomial Coeff} \\ &= \int \tan^8 x d(\tan(x)) + 2 \int \tan^6 x d(\tan(x)) + \int \tan^4 x d(\tan(x)) \\ &= \frac{\tan^9 x}{9} + \frac{2}{7} \tan^7(x) + \frac{\tan^5(x)}{5} + C. \end{aligned}$$

C. Don't know how to do it for now (exception: $\int \sec(x) dx \rightarrow (4)$)

$m=0, n=1$

③ Again : 3 cases + Trig identities

- A. m odd
- B. n even
- C. neither

$$\begin{cases} 1 + \cot^2(x) = \csc^2(x) \quad (\text{n even}) \\ \cot^2(x) = \csc^2(x) - 1 \quad (\text{m odd}) \\ d(\cot(x)) = -\csc^2(x) dx \\ d(\csc(x)) = -\cot(x) \csc(x) dx \end{cases}$$

Note: For ② & ③ if m odd & n even we can use any of the 2 methods!

Remark: Sometimes it's easier to express everything via $\sin(x)$ & $\cos(x)$.

Eg: $\int \sec(x) \tan(x) dx = \int \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} dx = \int \frac{\sin(x)}{\cos^2(x)} dx$

$$\stackrel{u=\cos(x)}{=} \int \frac{-du}{u^2} = u^{-1} + C = \frac{1}{\cos(x)} + C = \sec(x) + C \quad \text{which}$$

we already knew from $d(\sec(x)) = \sec(x) \tan(x) dx$.

Integration formulas

[C = constant]

$$\textcircled{1} \int u^n du = \frac{u^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\textcircled{2} \int u^{-1} du = \int \frac{1}{u} du = \ln u + C$$

$$\textcircled{3} \int e^u du = e^u + C$$

$$\textcircled{4} \int \cos u du = \sin u + C$$

$$\textcircled{5} \int \sin u du = -\cos u + C$$

$$\textcircled{6} \int \sec^2 u du = \tan u + C$$

$$\textcircled{7} \int \csc^2 u du = -\cot u + C$$

$$\textcircled{8} \int \sec u \tan u du = \sec u + C$$

$$\textcircled{9} \int \csc u \cot u du = -\csc u + C$$

$$\textcircled{10} \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \quad (x = \frac{u}{a} \text{ substitution})$$

$$\textcircled{11} \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \quad (x = \frac{u}{a} \text{ ---})$$

$$\textcircled{12} \int \tan(u) du = -\ln |\cos u| + C \quad (x = \cos u \text{ ---})$$

$$\textcircled{13} \int \cot(u) du = \ln |\sin(u)| + C \quad (x = \sin u \text{ ---})$$

$$\textcircled{14} \int \sec(u) du = \ln |\sec(u) + \tan(u)| + C$$

$$\textcircled{15} \int \csc(u) du = -\ln |\csc(u) + \cot(u)| + C$$