

(11)

Lecture XXXVI: § 10.4 Trigonometric Substitutions
§ 10.5 Completing the Square

§ 1 Trigonometric Substitutions

Used to complete integrals involving

$$(1) \sqrt{a^2 - x^2}$$

$$(2) \sqrt{a^2 + x^2}$$

$$(3) \sqrt{x^2 - a^2}$$

where $a > 0$ is constant.

There are 3 cases & we substitute x by a trig function

$$(1) x = a \sin(u) \Rightarrow \sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2 u)} = \sqrt{a^2 \cos^2 u} = a \cos(u)$$

$$(2) x = a \tan(u) \Rightarrow \sqrt{a^2 + x^2} = \sqrt{a^2(1 + \tan^2 u)} = \sqrt{a^2 \sec^2 u} = a \sec(u)$$

$$(3) x = a \sec(u) \Rightarrow \sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 u - 1)} = \sqrt{a^2 \tan^2 u} = a \tan(u)$$

Examples:

$$\text{Type (1)} \quad x = a \sin(u) \quad dx = a \cos(u) du$$

$$(1) \int \frac{\sqrt{a^2 - x^2}}{x} dx \stackrel{x=a\sin(u)}{=} \int \frac{a \cos^2(u)}{a \sin(u)} du = a \int \frac{\cos^2 u}{\sin u} du = a \int \frac{1 - \sin^2 u}{\sin u} du = a \int \frac{1}{\sin u} du - a \int \sin u du$$

$$= a \int \frac{1}{\sin u} du + a \cos u + C \stackrel{\substack{\uparrow \\ \text{"csc}(u)}}{=} -a \ln(\csc(u) + \cot(u)) + a \cos u + C$$

$$\text{In terms of } x? \quad \csc(u) = \frac{1}{\sin u} = \frac{a}{x}, \quad \cot(u) = \frac{\cos u}{\sin u} = \frac{\sqrt{1 - \sin^2 u}}{\sin u} = \frac{\sqrt{1 - \frac{x^2}{a^2}}}{\frac{x}{a}} = \frac{\sqrt{a^2 - x^2}}{x}$$

$$\cos(u) = \sqrt{1 - \frac{x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\text{Conclusion: } \int \frac{\sqrt{a^2 - x^2}}{x} dx = -a \ln\left(\frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x}\right) + \sqrt{a^2 - x^2} + C$$

$$(2) \int \frac{dx}{x^2 \sqrt{a^2 + x^2}} \stackrel{x=a\tan(u)}{=} \int \frac{a \sec^2(u) du}{a^2 \tan^2(u) a \sec(u)} = \frac{1}{a^2} \int \frac{\sec(u) du}{\tan^2(u)}$$

$$x = a \tan(u) \quad dx = a \sec^2(u) du \quad = \frac{1}{a^2} \int \frac{1}{\sin u} \frac{\cos u}{\sin^2 u} du$$

$$= \frac{1}{a^2} \int \frac{\cos u}{\sin^2 u} du = \frac{1}{a^2} \cdot (\sin u)^{-1} + C$$

Again: write $\sin u$ in terms of x .

$$a \sec(u) = \sqrt{a^2 + x^2}$$

$$\cos(u) = a / \sqrt{a^2 + x^2} = \frac{1}{\sqrt{1 + \frac{x^2}{a^2}}}$$

$$\sin u = \sqrt{1 - \cos^2 u} = \sqrt{1 - \frac{1}{1 + \frac{x^2}{a^2}}} = \frac{x}{\sqrt{a^2 + x^2}}$$

Answer: $\int \frac{dx}{x^2 \sqrt{a^2 + x^2}} = -\frac{1}{a^2} \left[\left(\frac{x}{\sqrt{a^2 + x^2}} \right)^{-1} + C \right] = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C$

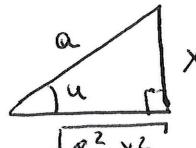
(Can check our solution!)

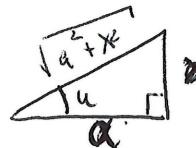
$$\left(-\frac{\sqrt{a^2 + x^2}}{a^2 x} \right)' = \frac{-x a^2 x}{a^4 x^2 \sqrt{a^2 + x^2}} + a^2 \frac{a^2 + x^2}{a^2 x^2 \sqrt{a^2 + x^2}} = \frac{-x^2 + a^2 + x^2}{a^2 x^2 \sqrt{a^2 + x^2}}$$

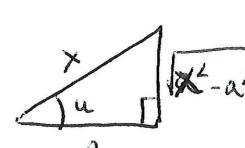
(3) $\int \frac{\sqrt{x^2 - a^2}}{x} dx \stackrel{\text{Type (3)}}{=} \int \frac{a \tan u}{a \sec u} a \sec u \tan u du = \int a \tan^2 u du$

$$= \int a (\sec^2 u - 1) du = \int a \sec u du - \int a du = a \underline{\tan u} - au + C = x/a \quad \text{"tan"}(\frac{x}{a})$$

Geometric interpretations: sides of a right triangle!

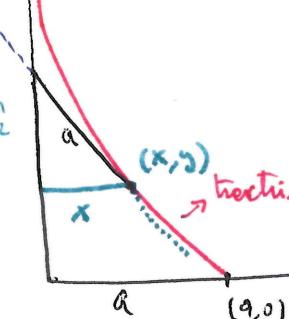
(1)  $x = a \sin u \Rightarrow \cos u = \frac{\sqrt{a^2 - x^2}}{a}$ & $\tan u = \frac{x}{\sqrt{a^2 - x^2}}$

(2)  $x = a \tan u \Rightarrow \cos u = \frac{a}{\sqrt{a^2 + x^2}}$ & $\sin u = \frac{x}{\sqrt{a^2 + x^2}}$

(3)  $x = \frac{a}{\cos u} = a \sec(u) \Rightarrow \cos u = \frac{a}{x}, \sin u = \frac{\sqrt{x^2 - a^2}}{x}$, $\tan u = \frac{x}{\sqrt{x^2 - a^2}}$

Application: tractrix curve.

- String of length a dragged along horiz plane
- move one end along the y -axis



At (x, y) the \overrightarrow{ab} is tangent to the curve:

$$\frac{dy}{dx} = \text{slope} = \frac{y}{x} \sqrt{a^2 - x^2} \Rightarrow \text{separate variables: } dy = \frac{\sqrt{a^2 - x^2}}{x} dx$$

Integrate $y(x) = y(x) - y(a) = - \int_a^x \frac{\sqrt{a^2 - t^2}}{t} dt = -\sqrt{a^2 - x^2} + a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) + \left(0 - a \ln \left(\frac{a}{a} \right) \right)$

$$y(a) = 0$$

Eqn: $y = a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) - \sqrt{a^2 - x^2}$

§2 Completing the square

Integrals involving $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ → to use §1, we need to complete the square!

Idea: $(x+A)^2 = x^2 + 2Ax + A^2$ $(\frac{x}{2})^2$

Eg: $5+4x-x^2 = -(x^2 - 4x - 3) = -(x^2 - 4x + 4 - 4 - 3)$
 $= -(x-2)^2 + 9.$
 $= 9 - (x-2)^2$

so $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-(x-2)^2}}$ $\uparrow u=x-2$ $= \int \frac{du}{\sqrt{9-u^2}}$

& now we can use the earlier method : Type (1) $\rightarrow a=3$.

Example 2: $\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{9+(x+1)^2} \stackrel{u=x+1}{=} \int \frac{du}{9+u^2} = \frac{1}{9} \tan^{-1}\left(\frac{u}{9}\right) + C$

$$x^2+2x+10 = x^2+2x+1-1+10 = (x+1)^2+9$$

Example 3: $\int \frac{x dx}{\sqrt{x^2-2x+5}} = \int \frac{x dx}{\sqrt{4+(x-1)^2}} \stackrel{u=x-1}{=} \int \frac{u+1 du}{\sqrt{4+u^2}}$ → Type (2) $\rightarrow a=2$

$$x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4$$

Example 4: $\int \frac{dx}{\sqrt{x^2-4x+5}} = \int \frac{dx}{\sqrt{1+(x-2)^2}} \stackrel{u=x-2}{=} \int \frac{du}{\sqrt{1+u^2}}$ is of Type (2)
 $\rightarrow a=1$

$$x^2-4x+5 = x^2-4x+4-4+5 = (x-2)^2+1$$

Example 5 $\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(x-2) + C$
 $(x^2-4x+3) = -(x^2-4x+4-4+3) = -((x-2)^2-1) = 1-(x-2)^2$

In general: $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$= a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4c-ab^2}{4a^2}\right)$$

$$= 0, > 0 \text{ or } < 0$$

Depending on the sign of $4c-ab^2$ we set a type (I), (II) or (III) integral.