

Lecture XXXVII: § 10.6 Partial Fractions
 § 10.7 Integration by Parts

Recall: So far we have learned

① Substitution: $\int f(g(x)) g'(x) dx = \int f(u) du$.

$$u = g(x)$$

② Special case: trigonometric substitution

$$a^2 - x^2 \Rightarrow x = a \sin \theta$$

$$x^2 + a^2 \Rightarrow x = a \tan \theta$$

$$x^2 - a^2 \Rightarrow x = a \sec \theta$$

For a general quadratic polynomial as complete squares!

TODAY: Partial fractions & integration by parts

$$\int \frac{P(x)}{Q(x)} dx = ? \quad \text{for } P, Q \text{ polynomials in } \mathbb{R}[x].$$

§ 1 STEP 1: Division of Polynomials

Division: $\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)}$

quotient
(a polynomial)

Q: How to find P_1 & R ? Long division!

$R = \text{remainder}$
 degree of $R < \text{degree of } Q$

with this choice, P_1 & R are unique

Example: $\frac{x^3 - 3x^2}{x^2 + 1} = P_1 + \frac{R}{x^2 + 1}$

$$\begin{array}{r}
 \overline{x^3 - 3x^2} \\
 \underline{-} \overline{x^3 + x} \\
 \hline
 -3x^2 - x \\
 - \overline{-3x^2 - 3} \\
 \hline
 \boxed{-x + 3} = R
 \end{array}$$

$$\hookrightarrow \text{degree} = 1 < 2 = \text{deg}(Q)$$

LONG DIVISION

- Take leading terms & write ratios in P
 $x = \frac{x^3}{x^2}, -3 = \frac{-3x^2}{x^2}$
 - Multiply ratio by Q & subtract it from P
 - Repeat with the new polynomial until its degree is < degree of Q.
- { The left over is the remainder R
 { The sum of the ratios is P_1

So $\frac{x^3 - 3x^2}{x^2 + 1} = x - 3 + \frac{(-x + 3)}{x^2 + 1}$

Example 2: $\frac{x^5 + 2x + 1}{x^3 - 2} = x^2 + \frac{zx^2 + 2x + 1}{x^3 - 2}$

$$\begin{array}{r} x^5 + 2x + 1 \\ \underline{- x^5 - 2x^2} \\ 2x^2 + 2x + 1 \end{array}$$

Conclusion: Using the polynomial division, we've reduced the integration problem

to $\int \frac{R(x)}{Q(x)} dx$ where $\deg(R) < \deg(Q)$

FACT: Every polynomial over \mathbb{R} = product of linear & quadratic polys
 $\begin{matrix} \text{indegree} \\ (\deg 1) \end{matrix}$ $\begin{matrix} \text{indegree} \\ (\deg 2) \end{matrix}$
 $\left[\text{w/o real roots} \right]$

§ 2 Partial Fractions

IDEA: Write $\frac{R(x)}{Q(x)}$ with $\deg(R) < \deg(Q)$ as a sum of rational functions of the form $\frac{1}{(x-a)^m}$ & $\frac{Ax+B}{(ax^2+bx+c)^n}$, $n, r > 0$ integers.

Ex 1: $\frac{12x-7}{(x-1)(x-2)} = \frac{-5}{x-1} + \frac{17}{x-2}$ (reverse the process of getting common denominator)

so $\int \frac{12x-7}{(x-1)(x-2)} dx = -5 \ln|x-1| + 17 \ln|x-2| + C$

There are 2 cases to analyze, depending on the multiplicity & nature of the factors of Q .

CASE 1: $Q(x) = (x-r_1)(x-r_2) \cdots (x-r_n)$ all real roots & all distinct.

We write $\frac{R(x)}{Q(x)} = \frac{A_1}{x-r_1} + \cdots + \frac{A_n}{x-r_n}$

We can find A_1, \dots, A_n by (1) evaluating at ^{n random} enough numbers ($\neq r_1, \dots, r_n$)

(2) Take common denominator & equate each coefficient in $R(x)$ to the numerator in the RHS

Ex: $\frac{9x^2+6}{x(x-2)(x-3)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x-3}$

$$\left. \begin{array}{l} \text{Method 1: } x=1: \frac{15}{1(-1)(1-2)} = A_1 - A_2 - \frac{A_3}{2} \\ x=-1: \frac{15}{-1(-3)(-4)} = -A_1 - \frac{A_2}{3} - \frac{A_3}{4} \\ x=-2: \frac{15}{-2(-4)(-5)} = \frac{A_1}{-2} - \frac{A_2}{4} - \frac{A_3}{5} \end{array} \right\} \quad \begin{array}{l} \text{3 linear equations in } A_1, A_2, A_3 \\ 15 = 2A_1 - 2A_2 - A_3 \\ 15 = 12A_1 + 4A_2 + 3A_3 \\ 15 = 20A_1 + 10A_2 + 8A_3 \end{array}$$

Method 2: Common denominator

$$\begin{aligned} \text{Num} &= A_1(x-2)(x-3) + A_2x(x-3) + A_3x(x-2) \quad (*) \\ &= A_1(x^2 - 5x + 6) + A_2(x^2 - 3x) + A_3(x^2 - 2x) \\ &= (A_1 + A_2 + A_3)x^2 + (-5A_1 - 3A_2 - 2A_3)x + (6A_1) = R_{(x)} = 9x^2 + 6 \\ \text{and } A_1 + A_2 + A_3 &= 9 \\ -5A_1 - 3A_2 - 2A_3 &= 0 \\ 6A_1 &= 6 \Rightarrow A_1 = 1 \end{aligned}$$

$$\begin{cases} 1 + A_2 + A_3 = 9 \\ -5 - 3A_2 - 2A_3 = 0 \end{cases} \quad \begin{array}{l} \text{replace in other} \\ 2 \text{ eqns} \end{array}$$

$$\begin{array}{l} \text{from } \begin{cases} A_2 + A_3 = 8 \Rightarrow A_2 = 8 - A_3 \\ -3A_2 - 2A_3 = 5 \end{cases} \\ \text{substitute } \\ -3(8 - A_3) - 2A_3 = 5 \\ -24 + 8A_3 = 5 \\ A_3 = 29 \end{array}$$

$$\text{so } A_3 = 29, \quad A_2 = 8 - 29 = -21$$

Easier method: write numerator as (*) & evaluate at r_1, \dots, r_n &
compare to $R(r_1), \dots, R(r_n)$

Advantage: At each evaluation only one term is nonzero:

$$\begin{array}{ll} \text{Eg: } r_1 = 0 & A_1(-2)(-3) + 0 + 0 = R(0) = 6 \Rightarrow 6A_1 = 6 \Rightarrow A_1 = 1 \\ r_2 = 2 & 0 + A_2 2(-1) + 0 = R(2) = 42 \Rightarrow -2A_2 = 42 \Rightarrow A_2 = -21 \\ r_3 = 3 & 0 + 0 + A_3 3(+1) = R(3) = 87 \Rightarrow 3A_3 = 87 \Rightarrow A_3 = 29 \end{array}$$

$$\text{Answer: } \int \frac{9x^2 + 6}{x(x-2)(x-3)} dx = \int \frac{1}{x} - \frac{21}{x-2} + \frac{29}{x-3} dx = \ln|x| - 21 \ln|x-2| + 29 \ln|x-3| + C$$

CASE 2: All real roots but with multiplicities, ie $(x-r_k)^m$ shows up in the factorization of Q & $m \geq 2$ integer.

Replace $\frac{A_k}{x-r_k}$ from Case 1 by $\frac{A_{k,1}}{(x-r_k)} + \frac{A_{k,2}}{(x-r_k)^2} + \dots + \frac{A_{k,m}}{(x-r_k)^m}$

→ As many summands for each real root as the multiplicity of the root!
 We use similar methods to find the constants A_{x_1}, \dots, A_{x_m} as that for Case 1.

Example $\frac{2x+1}{(x-1)^3} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3}$.

$$2x+1 = A_1(x-1)^2 + A_2(x-1) + A_3 \quad (*)$$

• Set $x=1 \Rightarrow 3 = 0 + 0 + A_3 \quad \text{so } A_3 = 3$

• Replace in (*) & factor out $(x-1)$

$$2x+1 - A_3 = 2x+1-3 = 2x-2 = 2(x-1) = A_1(x-1)^2 + A_2(x-1)$$

• Divide both sides by $(x-1)$: $2 = A_1(x-1) + A_2$

• Evaluate at $x=1$ & repeat until done: $\boxed{2 = A_2}$

Alternative: Evaluate at $x=1$, take derivatives up to order m & evaluate all at $x=1$ $2 - A_2 = 0 = A_1(x-1) \quad \text{so } A_1 = 0$

Answer: $\frac{2x+1}{(x-1)^3} = \frac{0}{(x-1)} + \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3} = \boxed{\frac{2}{(x-1)^2} + \frac{3}{(x-1)^3}}$

CASE 3: Quadratic factors with no real roots

If x^2+bx+c appears in Q with multiplicity = 1, then we have a summand $\frac{Ax+B}{x^2+bx+c}$ in the (RHS)

If $(x^2+bx+c)^m$ appears in Q with multiplicity $m \geq 2$, then we have in summands in the (RHS), namely

$$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \dots + \frac{A_mx+B_m}{(x^2+bx+c)^m}$$

We find the constants $A, B, A_1, B_1, \dots, A_m, B_m$ with the same methods as Cases 1 & 2 above.

Example ① $\int \frac{w \theta}{\sin^2 \theta - 5 \tan \theta + 4} d\theta = \int \frac{x dx}{x^2 - 8x + 4}$ $\stackrel{x = \tan \theta}{=} \int \frac{x dx}{(x-4)(x-1)} = \int \frac{x}{(x-4)(x-1)} dx$

$$\frac{x}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4} = \frac{A(x-4) + B(x-1)}{(x-1)(x-4)}$$

$$\text{At } x=1 : 1 = A(-3) = -3A \Rightarrow A = -\frac{1}{3}$$

$$\text{At } x=4 : 4 = B3 = 3B \Rightarrow B = \frac{4}{3}$$

$$\int \frac{x}{(x-4)(x-1)} dx = \int \frac{\frac{4}{3}}{x-4} - \frac{1}{3} \frac{1}{x-1} dx = \frac{4}{3} \ln(x-4) - \frac{1}{3} \ln(x-1) + C$$

② $\int \frac{x^2+2}{x^3+2x^2+2x} dx = \int \frac{x^2+2}{x(x^2+2x+2)} dx$ Quadratic formula
 $-2 \pm \sqrt{4-8} \over 2 \Rightarrow \text{no real roots!}$

$$\frac{x^2+2}{x(x^2+2x+2)} = \frac{A_1}{x} + \frac{A_2 x + B_2}{x^2+2x+2}$$

$$x^2+0x+2 \\ "x^2+2 = A_1(x^2+2x+2) + (A_2 x + B_2)x = (A_1+A_2)x^2 + (2A_1+B_2)x + 2A_1$$

Equate the 3 coefficients on each side

$$\begin{aligned} 1 &= A_1 + A_2 \rightarrow A_2 = 0 \\ 0 &= 2A_1 + B_2 \rightarrow 0 = 2 + B_2 \rightarrow B_2 = -2 \\ 2 &= 2A_1 \rightarrow A_1 = 1 \end{aligned}$$

$$\int \frac{x^2+2}{x^3+2x^2+2x} dx = \int \frac{1}{x} + \frac{(-2)}{x^2+2x+2} dx = \ln x - 2 \int \frac{1}{x^2+2x+2} dx$$

Complete squares: $x^2+2x+2 = (x+1)^2 - 1 + 2 = (x+1)^2 + 1$

$$= \ln x - 2 \int \frac{1}{(x+1)^2 + 1} dx = \ln x - 2 \tan^{-1}(u) + C = \ln x - 2 \tan^{-1}(x+1) + C.$$

Conclusion: $\int \frac{P(x)}{Q(x)} dx$ only as hard as computing $\int \frac{A}{(x-r)^P} dx$ (easy)

if $\int \frac{Ax+B}{(x^2+bx+c)^P} dx$ (substitution if $A \neq 0$)

. trig substitution if $A = 0$
after completing squares

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Example: $\frac{x+4}{x^2+2x+5}$

$$\frac{1}{dx} (x^2+2x+5) = 2x+2$$

$$x+4 = \frac{1}{2}(2x+2) + 3$$

$$\frac{x+4}{x^2+2x+5} = \frac{1}{2} \frac{2x+2}{x^2+2x+5} + \frac{3}{x^2+2x+5} \quad x^2+2x+5 = (x+1)^2 + 4$$

$$\int \frac{x+4}{x^2+2x+5} dx = \frac{1}{2} \int \frac{2x+2}{x^2+2x+5} dx + 3 \int \frac{1}{4+(x+1)^2} dx = \frac{1}{2} \ln(x^2+2x+5)$$

$$+ \frac{3}{4} \int \frac{dx}{1+(\frac{x+1}{2})^2} = \frac{1}{2} \ln(x^2+2x+5) + \frac{3}{4} \int \frac{2du}{1+u^2}$$

$$du = \frac{1}{2} dx$$

$$= \frac{1}{2} \ln(x^2+2x+5) + \frac{3}{2} \tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

§3 Integration by parts

Idea: Opposite of product rule

$$d(uv) = u dv + v du \quad \text{in integrate } uv = \int d(uv) = \int u dv + \int v du.$$

so $\boxed{\int udv = uv - \int v du}$

Key: find u & v so that integrating $\int v du$ is easier!

Example (1) $\int \underbrace{xe^x}_{=u} \underbrace{dx}_{dv} = xe^x - \int e^x dx = xe^x - e^x + C$

$$v = \int dv = \int e^x dx = e^x$$

(2) $\int \ln x \underbrace{\frac{1}{x} dx}_{dv} = (\ln x)x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 dx$
 $= x \ln x - x + C$

$$u = \int 1 dx = x \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

(3) [Recursion relation] $I_n = \int \underbrace{x^n e^x}_{=u} \underbrace{dx}_{dv} = x^n e^x - \int e^x du$
 $n \geq 1$ integer $= x^n e^x - \int e^x n x^{n-1} dx$

$$\Rightarrow I_n = x^n e^x - n I_{n-1} = x^n e^x - n (x^{n-1} e^x - (n-1) I_{n-2}) = \dots$$

$$(4) \int \underbrace{(\ln x)^2}_{=u} \underbrace{\frac{1}{x} dx}_{dv} = (\ln x)^2 x - \int 2 \ln x \cdot \frac{1}{x} \cdot x dx$$

$$= x(\ln x)^2 - 2 \int \ln x dx = x(\ln x)^2 - 2x \ln x + 2x + C$$

$$(5) \int \frac{1}{(x^2+a^2)^n} dx \quad n \geq 2$$

IDEA 1: Partial Fractions after integration by parts (To reduce the multiplicity)
 $d(x^2+a^2) = 2x dx$

$$\int \frac{1}{2x} \underbrace{\frac{2x dx}{(x^2+a^2)^n}}_{=dv} = \frac{1}{2x} \underbrace{\frac{1}{(-n+1)} \frac{1}{(x^2+a^2)^{n-1}}}_{=v} - \int \frac{-1}{2x^2} \cdot \frac{1}{n+1} \frac{1}{(x^2+a^2)^{n+1}} dx$$

$$= \frac{-1}{2(n+1)} \frac{1}{x} \frac{1}{(x^2+a^2)^{n-1}} - \frac{1}{2(n+1)} \int \frac{1}{x^2} \frac{1}{(x^2+a^2)^{n-1}} dx$$

$$\frac{1}{x^2 (x^2+a^2)^{n-1}} = \frac{A_0}{x} + \frac{B_0}{x^2} + \sum_{k=1}^{n-1} \frac{A_k x + B_k}{(x^2+a^2)^k}$$

So we need to find A_0, B_0, A_k, B_k for $k=1, \dots, n-1$ & use separation by parts to compute the remaining integrals when $A_k=0$.

IDEA 2: Use a trig substitution x^2+a^2 uses

$x = a \tan \theta$
$dx = a \sec^2 \theta d\theta$
$x^2+a^2 = a^2(1+\tan^2 \theta)$
$= a^2 \sec^2 \theta$

$$\int \frac{1}{(x^2+a^2)^n} dx = \int \frac{a \sec^2 \theta d\theta}{a^{2n} \sec^{2n}(\theta)}$$

$$= \frac{1}{a^{2n-1}} \int \frac{1}{\sec^{2n-2} \theta} d\theta = \frac{1}{a^{2n-1}} \int \omega^{\frac{2n-2}{2}} d\theta$$

Need to solve $\int \omega^{\frac{p}{2}} \theta d\theta = \int \underbrace{\omega^{\frac{p-1}{2}}(\theta)}_{=u} \underbrace{\omega \theta d\theta}_{=dv} = \omega^{\frac{p-1}{2}} \theta \sin \theta - \int v du$

$$I_p = \int \omega^{\frac{p}{2}} \theta d\theta = \omega^{\frac{p-1}{2}} \theta \sin \theta - \int \sin \theta (p-1) \omega^{\frac{p-2}{2}} (-\omega \theta) d\theta$$

$$= \omega^{\frac{p-1}{2}} \theta \sin \theta + (p-1) \int \omega^{\frac{p-2}{2}}(\theta) \underbrace{(\sin^2 \theta)}_{=1-\cos^2 \theta} d\theta$$

$$= \omega^{\frac{p-1}{2}} \theta \sin \theta + (p-1) \int \omega^{\frac{p-2}{2}} d\theta - (p-1) \int \omega^{\frac{p-2}{2}} \theta d\theta$$

$$\textcircled{I}_p = \omega^{\frac{p-1}{2}} \theta \sin \theta + (p-1) I_{p-2} - (p-1) \textcircled{I}_p$$

$$\Rightarrow \boxed{I_p = \frac{1}{p} \omega^{\frac{p-1}{2}} \theta \sin \theta + \frac{p-1}{p} I_{p-2}}$$

using recursive formula to find I_p .