

Lecture XXXVIII: §10.3 A mixed bag: integrals of miscellaneous types

Example: $J_p = \int \sin^p \theta d\theta$ for $p \geq 2$, $J_1 = \int \sin \theta d\theta = -\cos \theta + C$

$$J_p = \int \sin^{p-1} \theta \underbrace{\sin \theta d\theta}_{=dv} = \sin^{p-1}(\theta) (-\cos \theta) - \int (-\cos \theta) (p-1) \sin^{p-2} \theta \cos \theta d\theta$$

$$= -\sin^{p-1} \theta \cos \theta + (p-1) \int \underbrace{\cos^2 \theta}_{=1-\sin^2 \theta} \sin^{p-2} \theta d\theta = -\sin^{p-1} \theta \cos \theta + (p-1) \int \sin^{p-2} \theta d\theta - (p-1) \int \sin^p \theta d\theta$$

$$J_p = -\sin^{p-1} \theta \cos \theta + (p-1) J_{p-2} - (p-1) J_p \implies J_p = \frac{-\sin^{p-1} \theta \cos \theta}{p} + \frac{p-1}{p} J_{p-2}$$

NOTE: p & $p-2$ here have same parity!

Example $p=4$: $J_4 = \frac{-\sin^3 \theta \cos \theta}{4} + \frac{3}{4} J_2 = \frac{-1}{4} \sin^3 \theta \cos \theta + \frac{3}{4} \left(\frac{-\sin^2 \theta \cos \theta}{2} + \theta \right)$

$$J_2 = \int \sin^2 \theta d\theta = \int \sin \theta \cdot \frac{\sin \theta d\theta}{=d(-\cos \theta)} = -\sin \theta \cos \theta + \int \frac{\cos^2 \theta}{1-\sin^2 \theta} d\theta = -\sin \theta \cos \theta + \theta - \int \sin^2 \theta d\theta$$

$$\implies J_2 = \frac{-1}{2} \sin \theta \cos \theta + \theta$$

Alternative: (1) $\int \sin^{2p+1} \theta d\theta = \int \frac{\sin^{2p} \theta}{(\sin^2 \theta)^p} \sin \theta d\theta = \int (1-\cos^2 \theta)^p d(\cos \theta)$

$$= \int_{u=\cos \theta} (1-u^2)^p du = \int \sum_{k=0}^p \binom{p}{k} (-1)^k u^{2k} du = \sum_{k=0}^p \binom{p}{k} (-1)^k \frac{u^{2k+1}}{2k+1}$$

Binomial coeff.

$$= \sum_{k=0}^p \binom{p}{k} (-1)^k \frac{\cos^{2k+1}(\theta)}{2k+1}$$

(2) $\int \sin^{2p} \theta d\theta = \int \sin^{2p-1} \theta \sin \theta d\theta = -\sin^{2p-1} \theta \cos \theta - \int -\cos^2 \theta (2p-1) \sin^{2p-2} \theta d\theta$

$$J_{2p} = \frac{-\sin^{2p-1} \theta \cos \theta}{2p} + \frac{2p-1}{2p} J_{2(p-1)}$$

\implies recursive relation is the only option & we go all the way down to J_2

Techniques for integration \implies reduce to the 15 fundamental formulas

- substitution
- trig substitution ($x = a \sin \theta$, $x = a \tan \theta$, $x = a \sec \theta$)
($a^2 - x^2$) ($a^2 + x^2$) ($x^2 - a^2$)
- partial fractions & square completion
- integration by parts
- trig identities, simplifications

→ 3 more formulas to add to our list.

(16) $\int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx$
 $= \frac{1}{2a} (\ln(x-a) - \ln(x+a)) + C = \frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + C$

(17) $\int \frac{dx}{a^2 - x^2} = - \int \frac{dx}{x^2 - a^2} = -\frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + C = \frac{1}{2a} \ln \left(\frac{x+a}{x-a} \right) + C$

(18) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2 \theta d\theta}{a \sqrt{\sec^2 \theta}} = \int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + C$
 $= \ln \left(\sqrt{\frac{a^2 + x^2}{a^2}} + \frac{x}{a} \right) + C$
 $= \ln(\sqrt{a^2 + x^2} + x) + \underbrace{\ln a^{-1}}_{\tilde{C}} + C$
 $= \ln(\sqrt{a^2 + x^2} + x) + \tilde{C}$
 Substitutions:
 $x = a \tan \theta$
 $dx = a \sec^2 \theta d\theta$
 $x^2 + a^2 = a^2 \sec^2 \theta$

(18') $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} = \int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + C$
 $= \ln \left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right) + C$
 $= \ln(x + \sqrt{x^2 - a^2}) + \underbrace{\ln a^{-1}}_{\tilde{C}} + C$
 $= \ln(x + \sqrt{x^2 - a^2}) + \tilde{C}$
 Substitutions:
 $x = a \sec \theta$
 $dx = a \sec \theta \tan \theta d\theta$
 $\sqrt{x^2 - a^2} = a \tan \theta$

32 More examples:

Ex 1: $\int \frac{x^2}{x^6 - 1} dx$ Partial Fraction

$(x^6 - 1) = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x^2 - x + 1) = (x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1)$

$\frac{x^3 - 1}{x^3 - x^2} = \frac{x - 1}{x^2 + x + 1}$ → Quad. formula: $\frac{-1 \pm \sqrt{1 - 4}}{2}$ → no real roots

$\frac{x^2}{x^6 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 - x + 1} + \frac{Ex + F}{x^2 + x + 1}$

→ $x^2 = A(x + 1)(x^2 - x + 1)(x^2 + x + 1) + B(x - 1)(x^2 - x + 1)(x^2 + x + 1) + (Cx + D)(x^2 - 1)(x^2 + x + 1) + (Ex + F)(x^2 - 1)(x^2 - x + 1)$

$x = 1$: $1 = A \cdot 2 \cdot 1 \cdot 3 = 6A$, $x = -1$: $1 = B(-2) \cdot 3 \cdot 1 = -6B$ → $A = \frac{1}{6}$, $B = -\frac{1}{6}$
 $x = 0$: $0 = A - B - D - F = \frac{1}{6} - D - F$

$$x^2 = \frac{1}{6} (x^5 + x^3 + x + x^4 + x^2 + 1) - \frac{1}{6} (x^5 + x^3 + x - x^4 - x^2 - 1) + C(x^5 + x^4 - x^2 - x) + D(x^4 + x^3 - x - 1) + E(x^5 - x^4 + x^2 - x) + F(x^4 - x^3 + x - 1)$$

x^5 coeff: $0 = C + E \implies E = -C$

x^4 —: $0 = \frac{1}{3} + C + D - E + F = \frac{1}{3} + 2C + 2F = \frac{1}{3} - \frac{2}{3} + 2F = -\frac{1}{3} + 2F \checkmark$

x^3 —: $0 = D - F \implies D = F$

x^2 —: $1 = \frac{1}{3} - C + E = \frac{1}{3} - 2C \implies 2C = -\frac{2}{3} \rightarrow \boxed{C = -\frac{1}{3}} \leftarrow E = \frac{1}{3}$

x —: $0 = -C - D - E + F = 0 \checkmark$

1 —: $0 = \frac{1}{3} - D - F = \frac{1}{3} - 2F \implies \boxed{F = \frac{1}{6}} \Delta D = \frac{1}{6}$

$$\int \frac{x^2}{x^6-1} dx = \int \frac{1}{6} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx + \int \frac{(-\frac{1}{3}x + \frac{1}{6})}{x^2-x+1} dx + \int \frac{\frac{1}{3}x + \frac{1}{6}}{x^2+x+1} dx$$

$= \frac{1}{6} \ln \left(\frac{x-1}{x+1} \right) \quad \text{(I)} \quad \text{(II)}$

$x^2-x+1 = (x-\frac{1}{2})^2 + \frac{3}{4} \quad d(x^2-x+1) = 2x-1 \quad d(x^2+x+1) = 2x+1$

$\text{(I)}: \frac{-1}{6} \int \frac{2x-1}{x^2-x+1} dx = -\frac{1}{6} \int \frac{du}{u} = -\frac{1}{6} \ln(x^2-x+1)$

$\text{(II)}: \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx = \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \ln(x^2+x+1)$

Alternative: $x^6 = (x^3)^2 \implies u = x^3 \quad du = 3x^2 dx$

$$\int \frac{x^2}{x^6-1} dx = \int \frac{\frac{du}{3}}{u^2-1} = \frac{1}{3} \int \frac{du}{(u-1)(u+1)} = \frac{1}{6} \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du = \frac{1}{6} \ln \left(\frac{x^3-1}{x^3+1} \right) + C$$

Ex 2: $\int \frac{x^2}{1+x^2} dx \implies$ Long division $\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = 1 - \frac{1}{1+x^2}$

$= \int 1 - \frac{1}{1+x^2} dx = x - \tan^{-1}(x) + C.$

Ex 3: $\int \frac{e^{2x}}{e^x-1} dx \stackrel{u=e^x}{=} \int \frac{u du}{u-1} = \int \left(1 + \frac{1}{u-1} \right) du = u + \ln(u-1) + C = e^x + \ln(e^x-1) + C$

Ex 4: $\int \frac{dx}{x(\ln x)^2} \stackrel{u=\ln x}{=} \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\ln x} + C.$

Ex 5 $\int \frac{4x+1}{x^2+1} dx$ $d(x^2+1) = 2x dx$

$$= \int \frac{2(2x)dx}{x^2+1} + \int \frac{dx}{x^2+1} = 2 \int \frac{du}{u} + \tan^{-1}(x) = 2 \ln(x^2+1) + \tan^{-1}(x) + C$$

Ex 6 $\int \frac{x^5}{(1+x^2)^4} dx = \int \sum_{k=1}^4 \frac{A_k x + B_k}{(1+x^2)^k} dx$

$x^5 = \sum_{k=1}^4 (A_k x + B_k) (1+x^2)^{4-k}$ \rightarrow Expand and compare coefficients

Method 2: $u = 1+x^2$ $du = 2x dx$ $x^4 = (u-1)^2$

$$\int \frac{x^5}{(1+x^2)^4} dx = \frac{1}{2} \int \frac{(u-1)^2}{u^4} du = \frac{1}{2} \int \frac{u^2 - 2u + 1}{u^4} du = \frac{1}{2} \int (1 - \frac{2}{u^3} + \frac{1}{u^4}) du$$

$$= \frac{1}{2} (u + \frac{1}{u^2} - \frac{1}{3u^3}) + C = \frac{1}{2} (1+x^2 + \frac{1}{(1+x^2)^2} - \frac{1}{3(1+x^2)^3}) + C$$

Ex 7: $\int \frac{x dx}{\sqrt[3]{x+1}}$ \downarrow $u = x+1$

$$= \int u^{-1/3} (u-1) du = \int (u^{2/3} - u^{-1/3}) du = \frac{3}{5} u^{5/3} - \frac{3}{2} u^{2/3} + C$$

$$= \frac{3}{5} (x+1)^{5/3} - \frac{3}{2} (x+1)^{2/3} + C$$

Ex 8: $\int \sqrt{\frac{1+x}{1-x}} dx = \int \sqrt{\frac{1+x}{1-x}} \sqrt{\frac{1+x}{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x dx}{\sqrt{1-x^2}}$

$$= \sin^{-1}(x) - \frac{1}{2} \int \frac{du}{u^{1/2}} = \sin^{-1} x - \sqrt{u} + C = \sin^{-1} x - \sqrt{1-x^2} + C$$

$u = 1-x^2$
 $du = -2x dx$

Ex 9: $\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x} \frac{1-\cos x}{1-\cos x} dx = \int \frac{1-\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} dx$

$$- \int \frac{\cos x dx}{\sin^2 x} \downarrow \text{[Half Angle]}$$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{du}{u^2} = \frac{1}{\sin x} + \int \frac{1}{\sin^2 x} dx$$

$$= \frac{1}{\sin x} - \cot(x) + C$$

Ex 10: $\int e^{\sqrt{x}} dx$ $u = \sqrt{x}$ \downarrow $u = \sqrt{x}$ $du = \frac{1}{2\sqrt{x}} dx = \frac{1}{2u} dx$

$$= \int e^u \cdot 2u du = 2 \int u e^u du = 2 (u e^u - \int e^u du)$$

$$= 2 \sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}} + C$$