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Lecture XXXIX : § 12.1 The MVT revisited  
 § 12.2 L'Hopital's Rule. The indeterminate form  $\frac{0}{0}$

§1 Introduction: Recall  $f, g$  (continuous) functions on  $x=a$ , with  $\lim_{x \rightarrow a} f(x) = M$  &  $\lim_{x \rightarrow a} g(x) = N$   
 then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{M}{N}$  provided  $N \neq 0$

We cannot use this limit law to compute this limit when  $N=0$ . The ratio can have any behavior we want. If  $M \neq 0$ , then limits will be  $\pm\infty$  TODAY:  $\frac{0}{0}$ .

Ex 1  $\lim_{x \rightarrow 0} \frac{x}{x} = 1$ ,  $\lim_{x \rightarrow 0} \frac{x^2}{x} = 0$ ,  $\lim_{x \rightarrow 0} \frac{x}{x^2}$  does not exist 

$\lim_{x \rightarrow 0} \frac{x \sin(\frac{1}{x})}{x}$  oscillates wildly  $\rightarrow$  no limit!

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x}{x^2} &= +\infty \\ \lim_{x \rightarrow 0^-} \frac{x}{x^2} &= -\infty\end{aligned}$$

Ex 2 : [Rational functions]  $\lim_{x \rightarrow a} \frac{x^n + a_{n-1}x^{n-1} + \dots + a_0}{x^m + b_{m-1}x^{m-1} + \dots + b_0} = P(x)$

• If  $a$  is a root of  $Q(x)$ , then  $Q(x) = (x-a)^m \tilde{Q}(x)$  with  $\tilde{Q}(a) \neq 0, m \geq 1$ .

$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \lim_{x \rightarrow a} \frac{(x-a)^r \tilde{P}(x)}{(x-a)^m \tilde{Q}(x)} = \lim_{x \rightarrow a} (x-a)^{r-m} \frac{\tilde{P}(x)}{\tilde{Q}(x)}$  Write   
 $P(x) \sim \tilde{P}(x)$  if  $r \geq m$  and  $\tilde{P}(a) \neq 0$ .

The limit exists if and only if  $r-m \geq 0$ .

• If  $a$  is not a root of  $Q(x)$ , use the limit law.

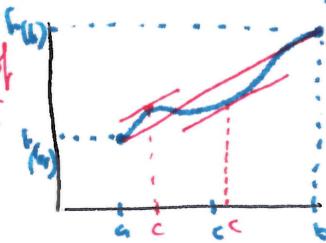
E 2  $\lim_{x \rightarrow 2} \frac{3x^2 - 7x + 2}{x^2 + 5x - 14} = \lim_{x \rightarrow 2} \frac{(x-2)(3x-1)}{(x-2)(x+7)} = \lim_{x \rightarrow 2} \frac{(3x-1)}{(x+7)} = \frac{5}{7}$ .

E 3 Geometry  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \sin'(0) = \cos(0) = 1$ .

In general:  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}$  is an indeterminate of the form  $\frac{0}{0}$  ( $= f'(a)$  if it exists!)

$\Rightarrow$  Expect a connection between derivatives &  $\frac{0}{0}$  indeterminates. (Use MVT)

Mean Value Thm: Given  $f: [a, b] \rightarrow \mathbb{R}$  continuous & differentiable on  $(a, b)$ , then there exists  $a < c < b$  with  $f'(c) = \frac{f(b) - f(a)}{b - a}$



Special case: Rolle's Thm: If  $f(a) = f(b)$ , we can

find  $c$  with  $f'(c) = 0$  (same hypothesis as MVT)

Often we generalize it?

Generalized MVT: Given  $f, g: [a, b] \rightarrow \mathbb{R}$  cont on  $[a, b]$  & diff'ble on  $(a, b)$ , then  
 There exists  $a < c < b$  with  $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$  with  $g'(x) \neq 0$  for all  $x \in (a, b)$ .

Proof: We can assume  $f(a) \neq f(b)$ , otherwise by Rolle's Thm  $f'(c) = 0$ , in contradiction with one of our hypothesis.

Consider the auxiliary function:

$$F(x) = (f(b) - f(a))(g(x) - g(a)) - (f(x) - f(a))(g(b) - g(a))$$

•  $F$  continuous on  $[a, b]$ , diff'ble on  $(a, b)$

$$\cdot F(a) = (f(b) - f(a)) \cdot 0 = 0 \quad (g(b) - g(a)) = 0$$

$$F(b) = (f(b) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(b) - g(a)) = 0 \quad \left. \right\} \text{ so } F(a) = F(b)$$

By Rolle's Thm applied to  $F$  we can find  $a < c < b$  with  $F'(c) = 0$ .

But  $F'(x) = (f(b) - f(a))(g'(x)) - f'(x)(g(b) - g(a))$  &  $F'(c) = 0$  is equiv.  
 to  $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$ .  $\square$

### §2 L'Hospital's Rule

L'Hospital's Thm: If  $f, g$  differentiable at  $x=a$ ,  $f(a) = g(a) = 0$  then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad \text{provided } g'(a) \neq 0$$

Proof: Write  $\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \xrightarrow{x \rightarrow a} \frac{f'(a)}{g'(a)}$  provided  $g'(a) \neq 0$

Examples ①  $\frac{f(x)}{g(x)} = \frac{\sin x}{x} \xrightarrow{x \rightarrow 0} \frac{(\sin x)'}{x'} \Big|_{x=0} = \frac{\cos(0)}{1} = 1$  b/c  $x' \Big|_{x=0} \neq 0$ .

②  $\frac{f(x)}{g(x)} = \frac{x^2}{x} \xrightarrow{x \rightarrow 0} \left(\frac{2x}{1}\right) \Big|_{x=0} = 0$ . % indet.

$$\begin{cases} \sin(0) = 0 \\ x(0) = 0 \end{cases}$$

③  $\lim_{x \rightarrow 0} \frac{\tan 6x}{e^{2x} - 1}$   $f(x) = \tan(6x)$   $f(0) = 0$   $f' = 6 \sec^2(6x)$   
 $g(x) = e^{2x} - 1$   $g(0) \neq 0$   $g'(x) = 2e^{2x}$   
 $g'(0) = 2 \neq 0$

By L'Hospital:  $\lim_{x \rightarrow 0} \frac{\tan 6x}{e^{2x} - 1} = \frac{6}{2} = \boxed{3}$

$$\begin{aligned} f'(0) &= 6 \sec^2(0) & f'(0) &= 6 \\ g'(0) &= 2e^{2 \cdot 0} & g'(0) &= 2 \neq 0 \end{aligned}$$

Q: What if we have  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \sim \frac{0}{0}$  AND  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \sim \frac{0}{0}$  ?

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We can think of it as an iteration, once we know both limits will agree.  
For this, we need Generalized MVT.

$$\text{Write } \frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x) - f(a)}{x-a}}{\frac{g(x) - g(a)}{x-a}} \stackrel{c \text{ in between } x \text{ & } a.}{\downarrow} \text{ for } \begin{cases} c \text{ in between } x \text{ & } a. \\ \text{[N.B. } g'(c) \neq 0] \end{cases}$$

As  $x \rightarrow a$ , we set  $c \rightarrow a$  so  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  if  $f'(a) = 0$   
& we can apply Generalized MVT. (ie the hypothesis are satisfied)

After this, we can repeat until some  $g^{(k)}(a) \neq 0$  & all previous  $\lim_{x \rightarrow a} \frac{f^{(j)}(x)}{g^{(j)}(x)} \sim \frac{0}{0}$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \dots = \lim_{x \rightarrow a} \frac{f^{(k)}(x)}{g^{(k)}(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}$$

% % ...

Theorem L'Hospital: Fix  $a$  in IR  
Pick  $f, g$  differentiable on some open interval containing  $a$ .  
Assume that  $g'(x) \neq 0$  in this interval except perhaps at  $x=a$ . If  $f(a)=g(a)=0$   
then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  provided the (RHS) limit exists.

Proof: Above discussion.

Ex 1:  $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} \sim \frac{0}{0}$        $f(x) = 1 - \cos(x)$        $f'(x) = \sin(x)$        $f(0) = g(0) = 0$   
 $g(x) = x^2$        $g'(x) = 2x$  not  $= 0$  for  $x \neq 0$

$$\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} \stackrel{\text{L'Hosp.}}{=} \lim_{x \rightarrow 0} \frac{\sin(x)}{2x} \sim \frac{0}{0}$$

$\stackrel{\text{L'Hosp.} \leftarrow 1}{}$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{2x} = \frac{\sin(0)}{2} = \boxed{\frac{1}{2}}$$
 This limit exists.

so all the previous limits also exists

Ex 2:  $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} \stackrel{\%}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} \stackrel{\text{L'Hosp.}}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{6x} \stackrel{\%}{=} \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \boxed{\frac{-1}{6}}$

$\boxed{3x^2 \neq 0 \text{ if } x \neq 0}$

$\boxed{6x \neq 0 \text{ if } x \neq 0}$

$\boxed{6 \neq 0 \text{ if } x \neq 0}$

Ex 3:  $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - (1 + \frac{1}{2}x)}{x^2} \stackrel{\%}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{x+1}} - \frac{1}{2}}{2x} \stackrel{\text{L'Hosp.}}{=} \lim_{x \rightarrow 0} \frac{-\frac{1}{4}\frac{1}{(x+1)^{3/2}}}{2} = \boxed{-\frac{1}{8}}$

$\boxed{2x \neq 0 \text{ if } x \neq 0}$

Warning: Need to verify the conditions  $\frac{0}{0}$ !

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{2x+3} = \frac{\sin 0}{3} = 0 \quad \text{Vs. } \lim_{x \rightarrow 0} \frac{(x\sin(4x))'}{(2x+3)'} = \lim_{x \rightarrow 0} \frac{4\cos(4x)}{2} = \frac{4}{2} = 2$$

What if  $x \rightarrow \infty$ ? We can do a change of variables  $x = \frac{1}{t}$ . If  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \equiv \frac{0}{0}$

- $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{t \rightarrow 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} \stackrel{\text{Chain Rule}}{\equiv} \lim_{t \rightarrow 0^+} \frac{f'(\frac{1}{t}) \cdot x'(\frac{1}{t})}{g'(\frac{1}{t}) \cdot x'(\frac{1}{t})} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$

- $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \lim_{t \rightarrow 0^-} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} \stackrel{\text{L'Hosp.}}{\equiv} \lim_{t \rightarrow 0^-} \frac{f'(\frac{1}{t}) \cdot x'(\frac{1}{t})}{g'(\frac{1}{t}) \cdot x'(\frac{1}{t})} = \lim_{x \rightarrow -\infty} \frac{f'(x)}{g'(x)}$

Conclusion: L'Hopital also applies when  $a = \pm\infty$ .

Ex:  $\lim_{x \rightarrow \infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}}$

$$\stackrel{\%}{=} \lim_{x \rightarrow \infty} \frac{\cos(\frac{1}{x})(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right)$$

$\downarrow$

$$\stackrel{\%}{=} \lim_{t \rightarrow 0^+} \cos(t) = \cos(0) = 1$$

This limit does exist!

Next Time: Other forms of indeterminates (best example =  $x \sin \frac{1}{x} \sim \infty \cdot 0$  for  $x \rightarrow \infty$ )