

Lecture XL: § 12.3 Other indeterminate forms

Recall: L'Hospital Thm: f, g differentiable near $x=a$, $g'(x) \neq 0$ for all $x \neq a$.
 & $f(a) = g(a) = 0$. Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

(either neither of them exists, or both do & they = each other).

Remark 1: We can extend this result to $x = +\infty$ or $x = -\infty$, via the change of variables $x = \frac{1}{t}$ & $t \rightarrow 0^+$, resp. $t \rightarrow 0^-$.

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} &= \lim_{t \rightarrow 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim_{t \rightarrow 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} \stackrel{\substack{\text{Chain Rule} \\ +}}{=} \lim_{t \rightarrow 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{t \rightarrow 0^+} \frac{f'(\frac{1}{t})}{x'(t)} = \lim_{t \rightarrow 0^+} \frac{f'(\frac{1}{t})}{\frac{1}{t'}} = \\ &= \lim_{x \rightarrow +\infty} \frac{f'(x)}{g'(x)} \quad \boxed{\text{L'Hosp. } g'(\frac{1}{t}) \neq 0 \text{ for } t \neq 0 \text{ & } x'(t) \neq 0 \text{ for } t \neq 0} \end{aligned}$$

Same process for $x \rightarrow -\infty$ & $t \rightarrow 0^-$ proves $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow -\infty} \frac{f'(x)}{g'(x)}$

$$\begin{aligned} \text{Ex: } \lim_{x \rightarrow -\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} &\stackrel{\substack{\text{L'Hosp.} \\ \uparrow \\ \downarrow}}{=} \lim_{x \rightarrow -\infty} \frac{\cos(\frac{1}{x})(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x \rightarrow -\infty} \cos(\frac{1}{x}) = \lim_{t \rightarrow 0^+} \cos(t) \\ (\text{Looks like } x \text{ vs } \frac{1}{x}) &\quad (\frac{-1}{x^2}) \neq 0 \quad \boxed{1} = \cos(0) \end{aligned}$$

TODAY'S GOAL: Extend this ideas to other indeterminates:

$0 \cdot \infty$, $\infty - \infty$, $\frac{\infty}{\infty}$, 0° , ∞° , 1^∞ . via algebraic manipulation.

CASE 1: $\frac{\infty}{\infty}$

Thm 1: If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \sim \frac{\infty}{\infty}$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ if the limit on the right exists (either in \mathbb{R} or $= \pm\infty$) & $g'(x) \neq 0$ for all $x \neq a$.

Proof: Suppose $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$. Want to show $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$.



Pick $\bar{x} < x < a$. By generalized MVT

$$\frac{f(x) - f(\bar{x})}{g(x) - g(\bar{x})} = \frac{f'(c)}{g'(c)} \text{ for some } \bar{x} < c < x$$

$$\frac{f'(c)}{g'(c)} \rightarrow L \text{ because } c \rightarrow a.$$

As $\bar{x} \rightarrow a$ we have

$$\text{Write } \frac{f(x) - f(\bar{x})}{g(x) - g(\bar{x})} = \frac{\frac{f(x)}{g(x)}}{\left(\frac{1 - \frac{f(\bar{x})}{f(x)}}{1 - \frac{g(\bar{x})}{g(x)}} \right)} = \frac{\frac{f'(c)}{g'(c)}}{\left(\frac{1 - \frac{f(\bar{x})}{f(x)}}{1 - \frac{g(\bar{x})}{g(x)}} \right)}$$

can be made
as close to L as
we want! □

• If \bar{x} is fixed near a , then $\frac{f(\bar{x})}{f(x)} \xrightarrow[x \rightarrow a]{} 0$ & $\frac{g(\bar{x})}{g(x)} \xrightarrow[x \rightarrow a]{} 0$

so $\lim_{x \rightarrow a^-} \frac{1 - \frac{f(\bar{x})}{f(x)}}{1 - \frac{g(\bar{x})}{g(x)}} = 1$ & we see that $\lim_{x \rightarrow a^-} \frac{f(x)}{g(x)} = \lim_{c \rightarrow a} \frac{f'(c)}{g'(c)} = L$ □

(They have the same limit since their ratio has limit = 1.)

Similar idea works for $\lim_{x \rightarrow a^+}$. a even if $L = \pm\infty$. □

Remark: The result is also valid for $a = \pm\infty$.

Example $E_p = \lim_{x \rightarrow \infty} \frac{x^p}{e^x} = \lim_{t \rightarrow 0^+} \frac{(\frac{1}{t})^p}{e^{1/t}} = \lim_{t \rightarrow 0^+} \frac{-p(\frac{1}{t})^{p+1}}{e^{1/t}(-\frac{1}{t^2})} = \lim_{t \rightarrow 0^+} \frac{(-p)(\frac{1}{t})^{p+1}}{e^{1/t}} =$

so $E_p = (-1)^p E_{p-1} = (-1)^p p! E_{p-2} = \dots = (-1)^k (p)(p-1)\dots(p-k+1) E_{p-k}$ for all $k < p$.

$$E_1 = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0.$$

so $E_p = (-1)^p p! E_1 = 0$ for all $p > 0$ integer.

Equivalently: $\lim_{x \rightarrow \infty} \frac{e^x}{x^p} = +\infty$ (e^x grows faster than any polynomial)

② $L_p = \lim_{x \rightarrow \infty} \frac{\ln x}{x^p} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{px^{p-1}} = \lim_{x \rightarrow \infty} \frac{1}{px^p} = 0$ for any p .

($\ln(x)$ grows slower than any polynomial)

CASE 2: Remaining indeterminate forms via algebraic manipulation

CASE A: $\frac{0 \cdot \infty}{f \cdot g}$ becomes $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by dividing by the reciprocal of one factor

Ex 1: $\lim_{x \rightarrow 0^+} \frac{x \ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$ or $\lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}}$

$$0 = \lim_{x \rightarrow 0^+} x = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}.$$

$\lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$ derivative is complicated!
= $-\frac{1}{x \ln^2 x}$

worse!

CASE B Remaining cases can be simplified by taking ln. or exp.

using $\ln(\lim_{x \rightarrow a} F(x)) = \lim_{x \rightarrow a} \ln F(x)$ (Bounded $F(x) > 0$ near a)

$\exp\left(\frac{\cdot}{\cdot}\right) = \lim_{x \rightarrow a} \exp(F(x))$

[\ln & \exp are continuous functions!] or taking common denominator ($\infty - \infty$)

Example 2: $\lim_{(0\infty-\infty)} \sec x - \tan x = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$

L'Hosp. $\stackrel{\infty - \infty}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \frac{0}{1} = \boxed{0}$

Example 3: $\lim_{(0^0)} x^x = ?$ Take $\ln!$ $\ln(\lim_{x \rightarrow 0^+} x^x) = \lim_{x \rightarrow 0^+} \ln(x^x)$

$\lim_{x \rightarrow 0^+} \ln(x^x) = \lim_{x \rightarrow 0^+} x \ln x = \boxed{0}$ so $\lim_{x \rightarrow 0^+} x^x = e^0 = \boxed{1}$.

Example 4 $\lim_{(\infty)} x^{1/x} = ?$ Take \ln .

$\ln(\lim_{x \rightarrow \infty} x^{1/x}) = \lim_{x \rightarrow \infty} \ln(x^{1/x}) = \lim_{x \rightarrow \infty} \frac{1}{x} \ln x \stackrel{\infty}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x} \ln x}{1} = \boxed{0}$

So $\lim_{x \rightarrow \infty} x^{1/x} = \exp(0) = e^0 = \boxed{1}$.

Example 5 $\lim_{(1^\infty)} (1+ax)^{1/x} = e^a$ for all a .

Take \ln . $\lim_{x \rightarrow \infty} \frac{1}{x} \ln(1+ax) \stackrel{\infty}{=} a$. $\frac{0}{0}$ indeterminate.

L'Hosp. $\stackrel{\infty}{=} \lim_{x \rightarrow 0} \frac{\frac{a}{1+ax}}{1} = \lim_{x \rightarrow 0} \frac{a}{1+ax} = a \quad \checkmark$

Recover the claim by exponentiation!

• Examples that take longer w/ L'Hospital:

① $\lim_{0} \frac{\sin^4 x}{x^4} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^4 = 1$ but will take 4 iterations of L'Hospital.

$$\textcircled{2} \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^2+1}{x^2}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{1}{x^2}} = \sqrt{1+0} = 1$$

but with L'Hopital $L = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+1}}$ so we haven't improved much!

③ Rational Functions: easier to factor than to use L'Hopital.

$$\textcircled{3} \lim_{x \rightarrow \infty} \frac{5x^4+x}{x^4+2x^2+1} = \lim_{x \rightarrow \infty} \frac{5 + \frac{x}{x^3}}{1 + \frac{2}{x^2} + \frac{1}{x^4}} = 5$$

divide by x^4

L'Hopital will take 4 iterations!