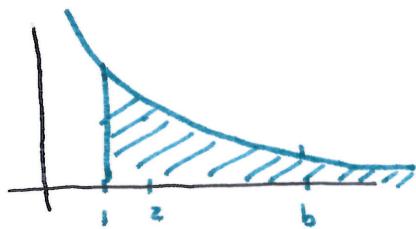


Lecture XL1 : §12.4 Improper integrals

GOAL: Compute $\int_a^b f(t) dt$ when: $a = -\infty$, and/or $b = +\infty$

$$\bullet \lim_{t \rightarrow a^+} f(t) = -\infty \text{ and/or } \lim_{t \rightarrow b^-} f(t) = +\infty$$

Examples: ① Calculate the area under the curve $y = \frac{1}{x^2}$ with $x \geq 1$.



$$\int_1^b \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_1^b = 1 - \frac{1}{b} \xrightarrow[b \rightarrow \infty]{=} 1 - 0 = 1$$

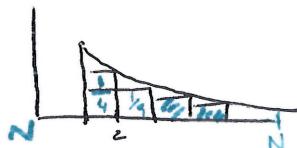
$$\text{Area} = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = 1 =: \int_1^{\infty} \frac{1}{x^2} dx$$

We say the integral converges

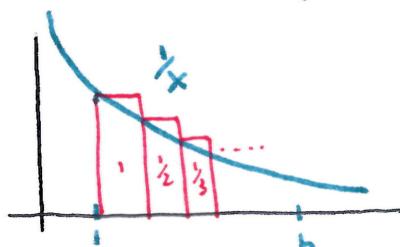
Note: Using Riemann sums. $x_n = \frac{1}{n+1}$

$$1 = \text{Area} \geq \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \left(\lim_{N \rightarrow \infty} \sum_{n=2}^N \frac{1}{n^2} \right) =: \sum_{n=2}^{\infty} \frac{1}{n^2}$$

We will use improper integrals to study series & their convergence.



②



$$\int_1^b \frac{1}{x} dx = \ln x \Big|_1^b = \ln b - \ln 1 = \ln b \xrightarrow[b \rightarrow \infty]{} \infty$$

the integral is divergent

So $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \not= \infty$
(Bounded below by Area)

③ In general

$$P \geq 0 \quad \int_1^b \frac{dx}{x^P} = \frac{x^{1-P}}{1-P} \Big|_1^b = \frac{1}{1-P} (b^{1-P} - 1) \xrightarrow[b \rightarrow \infty]{} \frac{1}{1-P}$$

$P \neq 1$

$$\Rightarrow \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^P} = \begin{cases} \frac{1}{1-P} & \text{if } P > 1 \\ \infty & \text{if } P \leq 1 \quad (\text{also for } P=1) \end{cases}$$

Def: $\int_a^{\infty} f(x) dx := \lim_{b \rightarrow \infty} \int_a^b f(x) dx$.

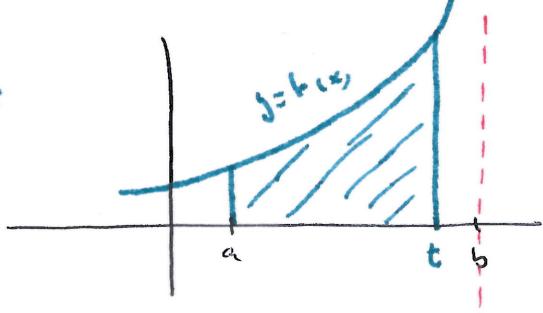
Applications: (1) Convergence/Divergence Series (Ch. 13) (2) LAPLACE TRANSFORM $L_f(P) = \int_0^{\infty} e^{-Px} f(x) dx$ for a given func f(x)

(3) GAMMA function: $\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1} dx$ using parts $\Gamma(p+1) = p \Gamma(p), \Gamma_p \text{ interpolates } (p-1)!$

(2)

Similar paradigm: Say $f(x)$ has a vertical asymptote at $x=b$

Def.



$$\text{Area under the curve} = \int_a^b f(x) dx$$

$$\int_a^b f(x) dx := \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

Note: The limit may or may not exist.

So the integral may converge or diverge.
(limit exists) (limit does not exist)
 $\Rightarrow \infty$

Examples: ① $\int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{\sqrt{1-x}} dx = \lim_{t \rightarrow 1^-} -2\sqrt{1-x} \Big|_0^t = \lim_{t \rightarrow 1^-} -2(\sqrt{1-t} - 1)$ converges

② $\int_0^t \frac{dx}{1-x} = \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{1-x} dx = \lim_{t \rightarrow 1^-} -\ln(1-x) \Big|_0^t = \lim_{t \rightarrow 1^-} -(\ln(1-t)) = \boxed{\infty}$ diverges

Similar behavior: $\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$.

$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$ ($x=a$ vertical asymptote)

Q: What about issues at both ends?

Use $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ for any $a < c < b$

The result should be independent of the choice of c . (If 2 integrals in the right converge, then the left one is defined.).

Typical choice $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$.

Eg: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow +\infty} \int_0^t \frac{1}{1+x^2} dx$ Term

$$+ \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} \tan^{-1} x \Big|_0^t + \lim_{t \rightarrow 0} \tan^{-1} x \Big|_0^t$$

$$= 0 - (-\frac{\pi}{2}) + (\frac{\pi}{2} - 0) = \boxed{\pi}$$

