

Lecture XLIV: § 13.2 Convergent sequence

§ 13.3: Convergent & divergent series

§ 1 Convergence criteria for sequences:

Thm 1: Assume $\{x_n\}_n$ is increasing ($x_n \leq x_{n+1}$ for all n). Then:

$\{x_n\}_n$ is convergent if and only if $\{x_n\}_n$ is bounded
(\Leftrightarrow)

We will prove this result by double implication. The direction (\Rightarrow) holds in general, so we write it as a separate lemma.

Lemma: If $\{x_n\}_n$ is convergent, then $\{x_n\}_n$ is bounded.

Proof: Write $\lim_{n \rightarrow \infty} x_n = L$. For $\epsilon = 1$, we can find n_0 such that

$$|x_n - L| < 1 = \epsilon \text{ for any } n \geq n_0, \text{ meaning } L-1 < x_n < L+1 \text{ for } n \geq n_0.$$

For $n < n_0$, need a different bound:

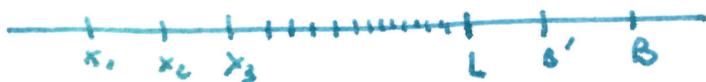
$$\left. \begin{aligned} \text{Pick } M &= \max \{x_1, x_2, \dots, x_{n_0-1}\} \\ N &= \min \{x_1, x_2, \dots, x_{n_0-1}\} \end{aligned} \right\} \Rightarrow M \leq x_n \leq N \text{ for } n=1, \dots, n_0-1$$

$$\left. \begin{aligned} \text{Take } B &= \max \{M, L+1\} \\ A &= \min \{N, L-1\} \end{aligned} \right\} \Rightarrow A \leq x_n \leq B \text{ for all } n, \text{ \& } \{x_n\}_n \text{ is bounded. } \square$$

Proof of Thm 1: By double implication.

(\Rightarrow) Is the statement of the Lemma

(\Leftarrow) Assume $\{x_n\}_n$ is bounded, we want to find the limit.



Pick $B \geq x_n$ for all n .

The smaller the B , the better the bound

We can find a least upper bound on \mathbb{R} by the way \mathbb{R} is constructed.

Least Upper Bound Axiom for \mathbb{R} : every non-empty set S in \mathbb{R} that has an upper bound also has a least upper bound ($= \inf \{B \text{ in } \mathbb{R} : x \leq B \text{ for all } x \text{ in } S\}$)

Set $L =$ least upper bound for $\{x_n\}_n$.

Claim: $L = \lim_{n \rightarrow \infty} x_n$.

\rightarrow tightest upper bound

[\mathbb{Q} does not have this property
 $S = \{x \text{ in } \mathbb{Q} : x < \sqrt{2}\}$ has upper bounds (eg $\sqrt{2}$) but $\text{LUB} = \sqrt{2}$ is irrational]

Why? By def of L.U.B, given $\epsilon > 0$, $L - \epsilon$ is no longer an U.B so

we can find x_{n_0} with $L - \epsilon < x_{n_0} \leq L$ Since x_n is increasing $L - \epsilon < x_{n_0} < x_n \leq L$ for all $n \geq n_0$

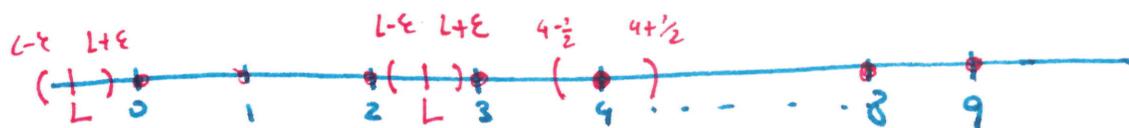
So $L - \epsilon < x_{n_0} \leq x_n \leq L$ for all $n \geq n_0$ gives $|x - L| < \epsilon$ if $n \geq n_0$ (2)

By def: $\lim_{n \rightarrow \infty} x_n = L$. \square

Application 1: $\{x_n\}$ where $x_n = n^{\text{th}}$ digit of π is bounded ($x_n \in \{0, 1, \dots, 9\}$)

Claim: The sequence has no limit & so it cannot be eventually increasing / decreasing (since this will contradict Thm 1 & Thm 2 from last lecture)

Q: Why does the sequence diverge? We look at the values of possible limits.



We claim no real number L can be $\lim_{n \rightarrow \infty} x_n$. We argue by contradiction:

CASE 1 L is not $0, 1, 2, \dots, 9$.

• For example, say $L < 0$ & pick $\epsilon = -\frac{L}{2} > 0$. Then $L + \epsilon = \frac{L}{2} < 0$ & so $x_n \notin (L - \epsilon, L + \epsilon)$ for all n , contradicting the def of limit of a sequence.

• Similar argument shows that if $L > 9$, then L can't be the limit of x_n .

• The remaining possibility is $j < L < j + 1$ for $j = 0, \dots, 8$. (in between 2 integers)

In this case, we can pick $\epsilon = \min \left\{ \frac{L - j}{2}, \frac{j + 1 - L}{2} \right\}$ & set:

$j < L - \epsilon < L < L + \epsilon < j + 1$, so again $0, 1, \dots, 9 \notin (L - \epsilon, L + \epsilon)$ so no x_n lies in $(L - \epsilon, L + \epsilon)$, contradicting the def. of limit of a sequence

CASE 2 L is one of $0, 1, \dots, 9$

For example, say $L = 4$. Pick $\epsilon = \frac{1}{2}$ so $(L - \epsilon, L + \epsilon)$ satisfies

$3 < 4 - \frac{1}{2} = L - \epsilon < L + \epsilon = 4 + \frac{1}{2} < 5$ & so the only value x_n we can have in $(L - \epsilon, L + \epsilon)$ is

$x_n = 4$. By the def. of limit we can find n_0 for which $|x_n - 4| < \epsilon = \frac{1}{2}$ for

all $n \geq n_0$. This forces $x_n = 4$ for all $n \geq n_0$, so the sequence is ultimately constant ($= 4$). Similar argument works for any $L \in \{0, \dots, 9\}$

What happens in this case? We write the decimal exp. of π .

$$\pi = \sum_{n=0}^{\infty} \frac{x_n}{10^n} = \underbrace{\sum_{n=0}^{n_0-1} \frac{x_n}{10^n}}_{= A \text{ in } \mathbb{Q}} + \sum_{n=n_0}^{\infty} \frac{L^{=x_n}}{10^n}$$

But $\sum_{n=n_0}^{\infty} \frac{L}{10^n} = \frac{L}{10^{n_0}} + \frac{L}{10^{n_0+1}} + \frac{L}{10^{n_0+2}} + \dots = \frac{L}{10^{n_0}} (1 + \frac{1}{10} + \frac{1}{10^2} + \dots + 1)$

So $\pi = A + \frac{L}{10^{n_0}} \left(\sum_{n=0}^{\infty} \frac{1}{10^n} \right) = A + \frac{L}{10^{n_0}} \cdot \frac{10}{9}$ in \mathbb{Q} .

But we know π is irrational! This gives the contradiction we wanted!

We conclude: $\{x_n\}$ has no limit.

NEXT TIME: $x_n = \left(1 + \frac{1}{n}\right)^n \rightarrow e$

§ 2. Convergent & divergent series:

Def: If a_1, a_2, a_3, \dots is a sequence, the serie with general term a_n is

the expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

Example (above) $\sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{1}{1} + \frac{1}{10} + \frac{1}{10^2} + \dots$ is a geometric series with general term $a_n = \frac{1}{10^n}$.

Attached to the series is another sequence: that of its partial sums

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \dots + a_n \end{aligned}$$

Eg above

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{10} \\ &\vdots \\ s_n &= 1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}} \end{aligned}$$

\uparrow
 $\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}$
 $= \frac{10}{9} \left(1 - \frac{1}{10^n}\right)$

We use the partial sums to define the sum of the series as a limit.

$$\sum_{j=1}^{\infty} a_j = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j = \lim_{n \rightarrow \infty} s_n$$

Eg (above) $s_n = \frac{10}{9} \left(1 - \frac{1}{10^n}\right) \xrightarrow{n \rightarrow \infty} \frac{10}{9}$

So the series converges to $\frac{10}{9}$!

Remark: If the limit of partial sums exists & it's finite ($\neq \pm \infty$ in \mathbb{R}) we say that the series converges to L (\Rightarrow that L is the sum of the series)

• If the partial sums have no limit or the limit is infinite ($= \pm \infty$), we say that the series diverges (\Rightarrow that the series is divergent).

Note: This is very similar to what we did with improper integrals!

$$\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t f(x) dx \quad \rightsquigarrow \quad \sum_{j=1}^{\infty} a_j = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j$$

We will use the connection to Riemann sums to compute various sums of series & to show other series diverge (eg: harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$)

Example: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ has partial sums $S_n = 2 \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) \xrightarrow{n \rightarrow \infty} 2(1-0) = 2$

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$

Q: What happens if we shift the indices:

E.g.: $\sum_{k=3}^{\infty} \frac{1}{2^k} = \frac{1}{2^3} + \frac{1}{2^4} + \dots$
 add 3 missing terms $1, \frac{1}{2}, \frac{1}{2^2}$
 \uparrow
 $= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots - \left(1 + \frac{1}{2} + \frac{1}{2^2}\right)$
 $= \left(\sum_{k=0}^{\infty} \frac{1}{2^k}\right) - \left(1 + \frac{1}{2} + \frac{1}{4}\right)$
 $= 2 - \frac{7}{4} = \boxed{\frac{1}{4}}$

• Alternative method: take common factor $\frac{1}{2^3}$.

$\Rightarrow \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots = \frac{1}{2^3} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) = \frac{1}{2^3} \sum_{k=0}^{\infty} \frac{1}{2^k}$
 $= \frac{1}{2^3} \cdot 2 = \frac{2}{8} = \boxed{\frac{1}{4}}$

Not surprisingly, we get the same answer!

Next: We generalize this series to other bases.

(\Leftarrow) Write $x = \frac{p}{q}$ & do the long division

$$q \overline{) \begin{array}{r} q_1 + q_2 + \dots \\ p \\ \underline{pq_1} \\ 0 < q - pq_1 < q \end{array}}$$

After some point we get remainder r with $0 \leq r < q$

If $r=0$, then x is an integer, so all decimals = 0.

If $r < q$, we continue.

$$q \overline{) \begin{array}{r} .r_1 \\ \underline{r_1 q} \\ r_2 q \\ \underline{ q} \\ \vdots \end{array}}$$

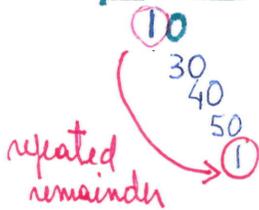
$0 \leq r_1 < q$

We look at remainders q_1, q_2, \dots

They are all in $\{0, 1, \dots, q-1\}$ so at some point we must repeat one.

From that point n , the ^{finite set} ratios r_i will start repeating ... \square

Example: $7 \overline{) 22} = 3.14571457\dots = 3.\overline{1457}$



$$\frac{22}{7} = 3.\overline{145714571457\dots} = 3.1457 \overline{1457}$$

$$= 3 + \underbrace{\left(\frac{1}{10} + \frac{4}{10^2} + \frac{5}{10^3} + \frac{7}{10^4}\right)}_{= \frac{1}{7}} \left(\underbrace{1 + \frac{1}{10^4} + \frac{1}{10^8} + \dots}_{\frac{10^4}{10^4-1}} \right)$$

Q: Manipulation of series?

1) $\sum_{n=0}^{\infty} a_n = s \implies$ Pick any $\lambda \in \mathbb{R}$ $\sum_{n=0}^{\infty} (\lambda a_n) = \lim_{n \rightarrow \infty} \sum_{j=0}^n \lambda a_j = \lim_{n \rightarrow \infty} \lambda \sum_{j=0}^n a_j$

$\stackrel{\text{Limit laws}}{=} \lambda \lim_{n \rightarrow \infty} \sum_{j=0}^n a_j = \lambda s.$

2) $\sum_{n=0}^{\infty} a_n = s, \sum_{n=0}^{\infty} b_n = t$. Then $\sum_{n=0}^{\infty} (a_n + b_n)$ converges & its sum is $s+t$.

Proof: $\sum_{n=0}^{\infty} a_n + b_n = \lim_{n \rightarrow \infty} \sum_{j=0}^n (a_j + b_j) = \lim_{n \rightarrow \infty} (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n)$

$\stackrel{\text{rearrange finite sum}}{=} \lim_{n \rightarrow \infty} ((a_1 + a_2 + \dots + a_n) + (b_1 + \dots + b_n)) \stackrel{\text{Limit laws}}{=} \lim_{n \rightarrow \infty} \sum_{j=0}^n a_j + \lim_{n \rightarrow \infty} \sum_{j=0}^n b_j = s + t \quad \checkmark$

Q: Can we always find a close formula for $s_n = \sum_{j=0}^n a_j$?

A: Not always. In these cases: just want to know convergence, but not the value of the sum.

Nice families: geometric \checkmark , telescopic have a close formula