

Lecture XLVII: § 13.5 Series of Non-negative terms. Comparison tests

Comparison Test 1: Assume $(a_n), (b_n)$ are two sequences of non-negative terms.

Assume $0 \leq a_n \leq b_n$ for all n . (for all $n \geq n_0$)

Then (1) If $\sum_{n=1}^{\infty} b_n$ converges to L , then $\sum_{n=1}^{\infty} a_n$ converges to $L' \leq L$
 $(L' \leq L + (a_1 - b_1) + \dots + (a_{n_0-1} - b_{n_0-1}))$

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

Example: (2) $\ln(n) < n$ for $n \geq 1$ (since $y = \ln(x)$ lies below the line $y = x$)

$$\text{so } \frac{1}{\ln(n)} > \frac{1}{n} > 0$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$.

(1) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ converges because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ does & $L' \leq \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

Proof: Assume $0 \leq a_n \leq b_n$ for all n . & take partial sums on both

sides $0 \leq \underbrace{a_1 + \dots + a_n}_{= s_n} \leq \underbrace{b_1 + \dots + b_n}_{= t_n}$

s_n & t_n are both increasing

so if $t_n \xrightarrow[n \rightarrow \infty]{} L$ we get $(s_n) \leq L$ for all n } $\lim_{n \rightarrow \infty} s_n = \sum_{n=1}^{\infty} a_n$ exists & $\leq L$.

If $s_n \xrightarrow[n \rightarrow \infty]{} \infty$ (because s_n is increasing & divergent, it can't be bounded)

then t_n is also not bounded & so $t_n \xrightarrow[n \rightarrow \infty]{} \infty$. □

We can generalize the result if $0 \leq a_n \leq b_n$ for all $n \geq n_0$ by rewriting both series by removing the first n_0 terms in both series:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + \dots + a_{n_0} + \boxed{\sum_{n=n_0+1}^{\infty} a_n} \\ \sum_{n=1}^{\infty} b_n &= b_1 + \dots + b_{n_0} + \boxed{\sum_{n=n_0+1}^{\infty} b_n} \end{aligned}$$

→ comparison test holds for the tail, so it holds for both series.

Example: $\sum_{n=1}^{\infty} \frac{n+1}{n^n}$ converges by comparing it to $\frac{2}{n^2}$.

$$\frac{n+1}{n^n} = \frac{1}{n^{n-1}} + \frac{1}{n^n} < \frac{2}{n^{n-1}} \leq \frac{2}{n^2} \text{ for } n \geq 3$$

$$\text{For } n \geq 3 \quad n-1 \geq 2 \quad \& \quad n^{n-1} \geq n^2$$

$$\text{so } \frac{1}{n^{n-1}} \leq \frac{1}{n^2} \quad \sum_{n=1}^{\infty} \frac{2}{n^2} \text{ converges, so does } \sum_{n=1}^{\infty} \frac{2}{n^{n-1}}$$

Limit Comparison Test: Suppose $(a_n), (b_n)$ non-negative terms, $b_n > 0$ for all n

Assume $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0$ (so $L > 0$). Then either both series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ converge or they both diverge

Why? If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ then for $\epsilon = \frac{L}{2}$ we can find N for which

$$\frac{a_n}{b_n} \in (L - \epsilon, L + \epsilon) = \left(\frac{L}{2}, \frac{3L}{2}\right) \Rightarrow \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$$

Since $b_n > 0$ we get $\frac{L}{2} b_n < a_n < \frac{3L}{2} b_n = d_n$

Since multiplying by $\frac{L}{2} \approx \frac{3L}{2}$ does not affect the convergence we conclude

- If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} \frac{3L}{2} b_n$. By comparison test for $0 \leq a_n < d_n$ we conclude $\sum_{n=1}^{\infty} a_n$ converges.

- If $\sum_{n=1}^{\infty} a_n$ converges, by comparison test for $0 \leq \frac{L}{2} b_n \leq a_n$, so does $\sum_{n=1}^{\infty} \frac{L}{2} b_n$. Hence, so does $\sum_{n=1}^{\infty} b_n$.

- If $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} \frac{L}{2} b_n$. By comparison test, so does $\sum_{n=1}^{\infty} a_n$.

- If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} \frac{3L}{2} b_n$ (by comparison test). Hence, so does $\sum_{n=1}^{\infty} b_n$.

Example: $\sum_{n=1}^{\infty} \frac{n+2}{2n^3 - 3}$ $a_n \geq 0$. $a_n = \frac{1+\frac{2}{n}}{2n^2 - \frac{3}{n}}$ compare it to $\frac{1}{n^2}$ or $\frac{1}{n^2 \cdot n}$
 since $\frac{a_n}{b_n} = \frac{1+\frac{2}{n}}{\frac{1}{n^2} - \frac{3}{n^3}} = n^2 \frac{1+\frac{2}{n}}{\frac{1}{n^2} - \frac{3}{n^3}} = \frac{1+\frac{2}{n}}{\frac{1}{n^2} - \frac{3}{n^3}} \xrightarrow{n \rightarrow \infty} \frac{1}{n^2} > 0$

Remark: Limit Comparison Test are useful if we compare with $\sum_{n=1}^{\infty} \frac{1}{n}, \sum_{n=1}^{\infty} \frac{1}{n^2}$ (but. if and only if $|x| < 1$) (DN)

Example 2 (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \leq 1$.

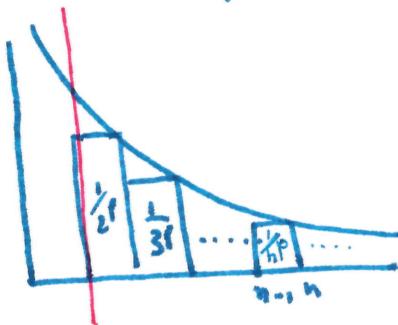
Why? If $p \leq 1$, then $n^p \leq n$ so $\frac{1}{n^p} \geq \frac{1}{n}$ by comparison test $\sum \frac{1}{n^p}$

If $p > 1$, will show that the partial sums sequence $(S_n)_n$ is diverges for $p \leq 1$ and bounded.

• If $p \geq 2$ we know it converges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$
because $\frac{1}{n^p} \leq \frac{1}{n^2}$ if $p \geq 2$.

• Missing cases $1 < p < 2$. 1

Method 1: Compare to area under the curve $f(x) = \frac{1}{x^p}$: [Cauchy Integral Test (review)]



$$-a_1 + s_n = \sum_{j=2}^n \frac{1}{j^p} < \int_1^\infty \frac{1}{x^p} dx = \left[\frac{x^{1-p}}{1-p} \right]_1^\infty = \frac{1}{1-p} < \infty$$

$$\text{so } s_n \leq \frac{1}{p-1} + a_1 \text{ for all } n, \quad \frac{1}{p-1} < \infty$$

so it converges (because $s_{n+1} = s_n + a_{n+1} > s_n > 0$ for all n)

Method 2 [Similar to bounding techniques we saw]

for $\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{1}{n!}$ series]

Figs n & show $s_n < \frac{2^{p-1}}{2^{p-1}-1}$.

How?? Pick m with $n < 2^m$. Then

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k^p} \leq \sum_{k=1}^{2^m} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^m-1)^p} \\ &= 1 + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p} \right)}_{\substack{\text{length term} \\ \text{2 terms}}} + \underbrace{\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right)}_{\substack{\text{(more terms!) length term} \\ 4 terms}} + \cdots + \underbrace{\left(\frac{1}{(2^{m-1})^p} + \cdots + \frac{1}{(2^m-1)^p} \right)}_{\substack{\text{largest term} \\ 2^{m-1} \text{ terms}}} \\ &\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{m-1}}{(2^{m-1})^p} \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \cdots + \frac{1}{(2^{m-1})^{p-1}} \end{aligned}$$

Write $a = \frac{1}{2^{p-1}}$:

$$= 1 + a + a^2 + \cdots + a^{m-1} \leq \sum_{n=1}^{\infty} a^n = \frac{1}{1-a} = \frac{2^{p-1}}{2^{p-1}-1}$$

Conclusion: $s_n < \frac{2^{p-1}}{2^{p-1}-1}$ for all n , so (s_n) is increasing & bounded
so it must converge & $\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{2^{p-1}}{2^{p-1}-1}$

Consequence of Comparison Test: Rearranging a convergent series of non-negative terms does not change its sum (Same for only negative terms)

Why? Say we rearrange a_n to b_n

$$\sum_n a_n = a_1 + a_2 + a_3 + a_4 + \dots = S \quad \text{all } s_n \text{ partial sums}$$

$$\sum_n b_n = a_{10} + a_3 + a_1 + a_{200} + \dots \quad \text{call } t_n \text{ partial sums}$$

$$\text{From here } t_4 = b_1 + b_2 + b_3 + b_4 = a_{10} + a_3 + a_1 + a_{200} \leq a_1 + a_2 + \dots + a_{200}$$

$$\text{So } t_4 \leq s_{200} \leq S \quad \begin{matrix} \uparrow \\ (a_n) \text{ all positive} \end{matrix} = s_{200}$$

Similarly all $t_n \leq S$. ($b_i = a_{m_i}$)

$$\begin{matrix} b_i = a_{m_i} \\ b_n = a_{m_n} \end{matrix} \left\{ \begin{matrix} \text{so } t_n \leq s_m \leq S \text{ for all } \\ m = \max\{m_1, \dots, m_n\} \end{matrix} \right.$$

• (b_n) is positive, so $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n = S$ gives $\sum_{n=1}^{\infty} b_n$ converges & sum $\leq S$

Symmetrically, since (a_n) is a rearrangement of a_n , the above arguments give $s = \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n \leq S$ so we conclude from here that $\sum_{n=1}^{\infty} b_n = S$ as well.

Warning: This fails if the series has alternating signs!

Example: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} a_n$

Multiply by $\frac{1}{2}$: $\frac{\ln 2}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots$

Add these 2 series term by term as follows (columnwise) & use formula for sum of this series!

$\ln 2 =$ $+ \quad - \frac{1}{2} \quad + \frac{1}{3} \quad - \frac{1}{4} \quad + \frac{1}{5} \quad - \frac{1}{6} \quad + \frac{1}{7} \quad - \frac{1}{8} \quad + \frac{1}{9} \quad - \frac{1}{10} \quad + \dots$
$\frac{\ln 2}{2} =$ $\underline{\underline{\frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \frac{1}{10}}}$

$$\ln 2 = \frac{1}{2} + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \dots = \sum_{n=1}^{\infty} b_n$$

(rearrangement of the original series with different sum!)