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Lecture XLVIII : §136 The Integral Test, Euler's Constant

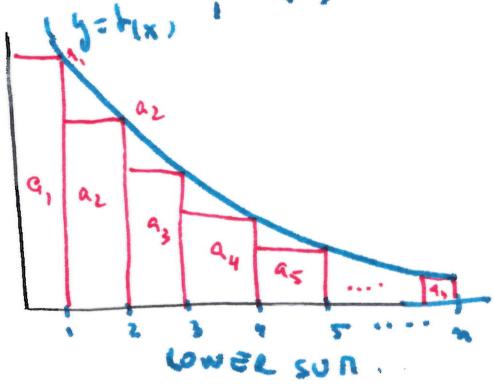
TODAY: Use improper integrals to test for convergence of $\sum_{n=1}^{\infty} a_n$.

What function do we use?

• Need $f: (0, \infty) \rightarrow \mathbb{R}$ with $f(n) = a_n$ & f continuous

• Need $a_n \geq 0$ for all n & (a_n) decreasing \Rightarrow Ask for f to satisfy this.

We'll relate $\int_1^{\infty} f(x) dx$ to lower & upper Riemann sums [$f' \leq 0$ \Leftrightarrow f decresc. f is decr.]



f decreasing and $f(x) \geq 0$ gives

$$\begin{aligned} a_2 + \dots + a_n &\leq \int_1^n f(x) dx \leq a_1 + \dots + a_{n-1} \\ (*) \quad \boxed{\sum_{j=1}^n a_j} &\leq a_1 + \int_1^n f(x) dx \leq a_1 + \int_1^{n+1} f(x) dx \leq a_1 + \boxed{\sum_{j=1}^n a_j} \end{aligned}$$

(Cauchy Integral Test)
Thm: Assume $f: [t, \infty) \rightarrow \mathbb{R}_{\geq 0}$ decreasing & $a_n = f(n)$ for all n .

Then either $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x) dx$ both converge or they both diverge.

Proof . . If $\int_1^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \int_1^n f(x) dx$ converges then $\sum_{j=1}^n a_j \leq a_1 + \int_1^{\infty} f(x) dx$ for all n , so the series converges (its partial sums are incr. & bounded). Similarly, if $\int_1^{\infty} f(x) dx$ diverges then $\sum_{j=1}^{\infty} a_j = \lim_{n \rightarrow \infty} \int_1^n f(x) dx \leq a_1 + \sum_{j=1}^{\infty} a_j$ so $\sum_{j=1}^{\infty} a_j$ diverges.

Inclusion: The series & the improper integral have the same behavior.

Applications Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Pick $f(x) = \frac{1}{x}$. Then $\begin{cases} f(n) = \frac{1}{n} = a_n \\ f \geq 0 \text{ & } f \text{ decr.} \\ f \text{ cont on } [1, \infty) \end{cases}$

$$\int_1^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \int_1^n \frac{1}{x} dx = \lim_{n \rightarrow \infty} \ln n - \underbrace{\ln(1)}_{=0} = \infty.$$

Example 2 (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Converges if and only if $p > 1$.
 (P=1 was previous example). Take $f(x) = \frac{1}{x^p}$

- f cont (& differentiable), $f \geq 0$ on $[1, \infty)$
- f decreasing because $f' = \frac{-p}{x^{p+1}} < 0$ on $[1, \infty)$.

$$\int_1^n f(x) dx = \int_1^n \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{1-p} \right]_1^n = \frac{1}{1-p} (n^{1-p} - 1)$$

$$\lim_{n \rightarrow \infty} \int_1^n f(x) dx = \frac{1}{1-p} \lim_{n \rightarrow \infty} (n^{1-p} - 1) = \begin{cases} \infty & 1-p > 0 \text{ dir} \\ \frac{1}{p-1} = \frac{1}{1-p} & 1-p < 0 \text{ up} \end{cases}$$

By Cauchy's Integral Test, the same is true for the p-series.

Example 3 $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ now $f(x) = \frac{1}{x \ln x}$ defined on $(2, \infty)$ & cont.

so f is decr. on $[2, \infty)$. Note: Cauchy's Integral Test will say. $\int_2^{\infty} f(x) dx$ does.

$$\int_2^n \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln n} u^{-1} du = \ln u \Big|_{\ln 2}^{\ln n} = \ln(\ln n) - \ln(\ln 2) \xrightarrow[n \rightarrow \infty]{\substack{\downarrow n \rightarrow \infty \\ \downarrow \infty}} \infty$$

so the series diverges.

Example 4: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ now $f(x) = \frac{1}{x(\ln x)^p}$ defined on $[2, \infty)$ & cont.

$$f' = \frac{-1}{(x(\ln x)^p)^2} ((\ln x)^p + p \ln(x)^{p-1}) = \frac{-(\ln x)^{p-1}(p + \ln x)}{x^2(\ln x)^{2p}} < 0 \text{ if } x > 2 \text{ so decreasing.}$$

We have $(\ln n)^p \leq \ln n$ for $p \leq 1$ & $n \geq 2$

so $\frac{1}{n(\ln n)^p} \geq \frac{1}{n \ln n}$ for $p \leq 1$ & $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, so

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ also diverges if $p \leq 1$.

For $p > 1$ we use Cauchy's Integral Test:

$$\int_2^n \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\ln n} \frac{1}{u^p} du = \frac{u^{1-p}}{1-p} \Big|_{\ln 2}^{\ln n} = \frac{1}{1-p} \left\{ (\ln n)^{1-p} - (\ln 2)^{1-p} \right\}$$

$$\infty = \frac{1}{1-p} \left(\frac{1}{(\ln n)^{p-1}} - \frac{1}{(\ln 2)^{p-1}} \right) > 0$$

$$\text{So } \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = 0 + \frac{1}{p-1} \frac{1}{\ln 2} \rightarrow p > 1 \quad \& \text{ by Leibniz's}$$

$$\text{Integral Test: } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \quad \sum_{n=2}^{\infty} \text{sum} \leq \frac{1}{2} \frac{1}{(\ln 2)^p} + \int_2^{\infty} \frac{1}{x \ln x} dx$$

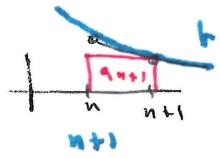
(Theorem gives an estimate for the sum!)

Q: What else can we learn from (*)?

Look at (*) & subtract $\int f(x) dx$ from all 3 terms, to get.

$$0 \leq \underbrace{\sum_{j=1}^n a_j - \int_1^n f(x) dx}_{(RHS)} \leq a_1$$

$=: F(n)$



We have: • F is bounded $0 \leq F(n) \leq a_1$.

• $F(n)$ is decreasing! $F(n+1) = F(n) + a_{n+1} - \int_n^{n+1} f(x) dx$
(b/c f is decreasing)

Then, the sequence $(F(n))_n$ has a limit L & $0 \leq L \leq a_1$.

$$L = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} \left(a_1 + \dots + a_n - \int_1^n f(x) dx \right)$$

Application: $a_n = \frac{1}{n}$ $f(x) = \ln x \Rightarrow \int_1^n \frac{1}{x} dx = \ln n$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \right) = L \quad \& \quad 0 \leq L \leq 1$$

Call it γ = Euler's Constant ($\approx 0.57721566490153286060\dots$, is $\gamma \in \mathbb{Q}$ or not?)

Rewrite lim as $1 + \frac{1}{2} + \dots + \frac{1}{n} - (\ln n + L) \xrightarrow{n \rightarrow \infty} 0$ in "algorithmic."

notation: $a_n = O(b_n)$ if $\frac{a_n}{b_n} \rightarrow 0$ $(\frac{1}{2} + \dots + \frac{1}{n} = \ln n + \gamma + O(1))$.

Consequence: Show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$.

$$\sum_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n} = \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) - \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right)$$

$$\stackrel{\substack{\text{odd terms} \\ \text{cancel}}}{=} \left(\gamma + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} \right) - 2 \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right) = \left(1 + \frac{1}{2} + \dots + \frac{1}{2n} \right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right)$$

$$\text{term} = \ln 2n + \gamma + O(1) - [\ln n + \gamma + O(1)] = \ln \frac{2n}{n} + O(1) = \ln 2 + O(1)$$

$$\text{so } \sum_{2n} \xrightarrow{n \rightarrow \infty} \ln 2$$

$$\bullet \text{Odd sums: } \sum_{2n+1} = \sum_{2n} + \frac{1}{2n+1} \xrightarrow{n \rightarrow \infty} \ln 2 + 0 \quad \checkmark$$