

# Lecture XLIX: The Ratio Test & the Root Test

## §1. The ratio test

Motivation: Convergence / divergence of geometric series with  $r > 0$

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \text{converges (to } \frac{1}{1-r} \text{)} & \text{if } r < 1 \\ \text{diverges (} = \infty \text{)} & \text{if } r > 1 \end{cases}$$

Idem: For the geom series, the ratio between successive terms is

the constant value  $r$ : 
$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r$$

So if  $r < 1$   $a_{n+1} = r a_n$  & the terms decay sufficiently fast to get convergence.

The ratio test says this always works for series of positive terms

Ratio Test: Pick a sequence  $(a_n)_{n \in \mathbb{N}}$  of positive terms and the associated series  $\sum_{n=1}^{\infty} a_n$ . Assume  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$  exists

Then (1) If  $L < 1$ , the series converges

(2) If  $L > 1$ , ——— diverges

(3) If  $L = 1$ , the test is inconclusive (anything can happen)

Examples: ①  $\sum_{k=0}^{\infty} \frac{1}{k!}$  we know this converges (to  $e$ )

The proof involved some careful bounding techniques.

The Ratio Test gives a much better proof:  $a_k = \frac{1}{k!} > 0$  for all  $k$ .

So 
$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{1}{k+1} \xrightarrow{k \rightarrow \infty} 0 = L < 1$$
 By Ratio Test the series converges.

②  $\sum_{k=1}^{\infty} \frac{3^k}{k!}$   $a_k = \frac{3^k}{k!} > 0$  for all  $k$  
$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}}{(k+1)!} = \frac{3}{k+1} \xrightarrow{k \rightarrow \infty} 0$$

so again converges by Ratio Test.

③  $\sum_{k=1}^{\infty} \frac{k^6}{3^k}$   $a_k = \frac{k^6}{3^k} > 0$  for all  $k$  
$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^6}{k^6} \frac{3^k}{3^{k+1}} = \left(\frac{k+1}{k}\right)^6 \frac{1}{3} \rightarrow \frac{1}{3}$$

Again, convergence follows from Ratio Test.

Note: Usefulness of Ratio Test if series involves factorials, powers, exp. products in general

④ Series  $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 5 \cdot 8} + \dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 5 \cdot 8 \dots (3n-1)}$

$$L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \dots 5 (2(n+1)-1) (2(n+1)-1)}{2 \cdot 5 \dots (3n-1) (3(n+1)-1)} = \lim_{n \rightarrow \infty} \frac{2(n+1)-1}{3(n+1)-1} = \frac{2}{3} < 1$$

so the series converges.

⑤  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  { converges for  $p > 1$   
 { diverges for  $0 < p < 1$

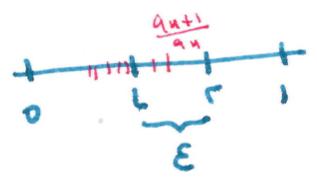
[p-series]  $\frac{a_{n+1}}{a_n} = \frac{1}{\frac{(n+1)^p}{n^p}} = \left(\frac{n}{n+1}\right)^p \rightarrow 1$

The test is inconclusive & anything can happen!

Proof of Ratio Test:

Idea: compare the geometric series  $\sum_{n=0}^{\infty} r^n$  (or a tail  $\sum_{n=n_0}^{\infty} r^n$  of it)

(1) Suppose  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$   $\Rightarrow$  Pick  $L < r < 1$ .



For  $\epsilon = r - L < 1$  we can find  $n_0$  for which  
 $(L - \epsilon <) \frac{a_{n+1}}{a_n} < L + \epsilon = r$  for all  $n \geq n_0$  (We know  $a_k > 0$  for all  $k$ )

so  $a_{n+1} < r a_n$   
 $a_{n_0+3} < r a_{n_0+1} < r^2 a_{n_0}$   
 $a_{n_0+3} < r a_{n_0+2} < r^3 a_{n_0}$   
 $\vdots$   
 $a_{n_0+k} < r^k a_{n_0}$

so  $\sum_{k=n_0+1}^{\infty} a_k < a_{n_0} \sum_{k=1}^{\infty} r^k < \infty$  so the series  $\sum_{k=n_0+1}^{\infty} a_k$  converges

b/c the bounded & positive, & so does  $\sum_{k=1}^{\infty} a_k$ . [Note: if  $\frac{a_{n+1}}{a_n} \rightarrow 1$  from above, same idea gives diverges. [AP A12 has refined test for  $\lim = 1$  from below]

(2) If  $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = L > 1$  then  $a_{k+1} > a_k > 0$  for all  $k \geq n_0$  so the terms are increasing & so  $a_n \not\rightarrow 0$ . The series diverges.

