

Lecture LII : §14.1 Introduction to power series

§14.2 The interval of convergence of a power series

GOAL : Understand expressions of functions in power series (w/ie

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$a_n = \text{constants}$   
 $x = \text{variable}$

(1) Given a known function (e.g. a soln to a differential eqn.), can we write it as a power series?

(2) If so, is the equality valid for any  $x$ ?

Examples : ①  $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$  for  $|x| < 1$ .

(note: cannot evaluate (RHS) at  $x=1$ )

②  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$       Region of validity = ??

(come from  $y' = y$  & trying to write a soln as  $\sum_{n=0}^{\infty} a_n x^n$ )  
 $y(0) = 1$       differentiate term by term to get  $a_n = \frac{1}{n!}$

③  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$       Region of validity = ??

[works for  $x=1$  (Lecture LI),  $x=0$ ]

(know  $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$  & integer power series)

④  $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^n$       Region of validity = ??

⑤  $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

Region of validity = ??

} Trying to solve  $y'' + y = 0$   
as power series, we set  
a series<sub>1</sub> + b series<sub>2</sub> =  $a \sin x + b \cos x$ .

Note: Given a power series  $\sum a_n x^n$ , whenever we have a <sup>specific</sup> value of the variable  $x$  for which a series converges, we can use it to define a function

(main = Region of validity in examples above).

The function may not be defined everywhere!

Example ① The geometric series  $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$  converges ABSOLUTELY

for  $|x| < 1$ . It diverges at  $x = \pm 1$  & for  $|x| > 1$ .

Conclusion: The geom series defines a function on  $(-1, 1)$ .

Note: In its domain, we know the function has the simpler formula  $\frac{1}{1-x}$

Example ②  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges ABSOLUTELY for all  $x$  [• For  $x=0$  sum =  $1+0+0+\dots=1$ ]

• For  $x \neq 0$ , show  $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$  converges by Ratio Test:  $\frac{a_{n+1}}{a_n} = \frac{|x|}{n+1} \rightarrow 0 < 1$

So Domain =  $\mathbb{R} = (-\infty, \infty)$ .

Example ③  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$  converges absolutely in  $(-1, 1)$

Why? For  $-1 < x < 1$  we get an alt. series. •  $a_n = \frac{x^n}{n} > 0$  for all  $n$ .  
 •  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  ( $x < 1$  so  $x^k \rightarrow 0$  as  $k \rightarrow \infty$ )  
 •  $(a_n)$  is ultimately decreasing!

By AST, the series converges.

• For  $-1 < x < 0$ :  $\sum_{k=1}^{\infty} \frac{(-1)(-x)^k}{k} = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$   
 $20 < \sum_{k=1}^{\infty} \frac{(-x)^k}{k} \leq \sum_{k=1}^{\infty} (-x)^k = \frac{1}{1+x} < \infty$  so it converges by comparison test!

• Diverges at  $x = -1$  because we get  $-\sum_{k=1}^{\infty} \frac{1}{k}$  (harmonic series)

• Converges at  $x = 1$  " " "  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

So Domain includes  $(-1, 1]$

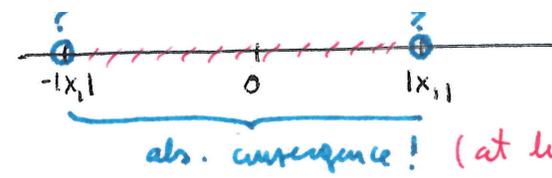
Q: What happens on  $(-\infty, -1)$ ?  $\sum_{k=1}^{\infty} \frac{1}{k}$  (harmonic series) diverges because series  $-\sum_{k=1}^{\infty} \frac{|x|^k}{k}$  &  $\frac{|x|^k}{k} \rightarrow \infty$

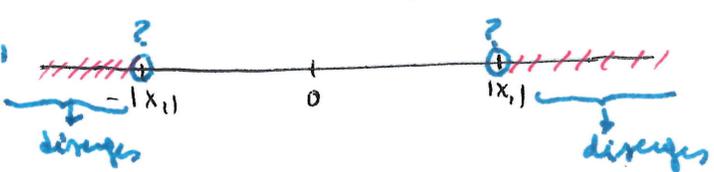
because  $(-1)^{n+1} \frac{x^n}{n} = A_n \rightarrow +\infty$  as  $n \rightarrow \infty$  ( $\frac{x^n}{n}$  behaves like  $\frac{e^n}{n}$ )

Q: To what extent are these ranges of convergence typical?

Useful Lemma: (i) If a power series  $\sum a_n x^n$  converges at  $x_1$  &  $x_1 \neq 0$ , then it converges absolutely for all  $x$  with  $|x| < |x_1|$ .

(ii) If the power series diverges at  $x_1$ , then it diverges for all  $x$  with  $|x| > |x_1|$

- Converges at  $x_1$ ,  (degrees  $|x_1|$  &  $-|x_1|$ )

- Diverges at  $x_1$ ,  (\_\_\_\_\_)

Why? (1) Suppose  $\sum_n a_n x^n$  converges at  $x = x_1$ .

Then  $a_n x_1^n \xrightarrow{n \rightarrow \infty} 0$  (the series converges!) so eventually, say  $\forall n \geq n_0$

We must have  $|a_n x_1^n| < 1$  (Pick  $\epsilon = 1$ )

But then, if  $|x| < |x_1|$ :  $|a_n x^n| = |a_n (\frac{x}{x_1})^n x_1^n| = |a_n x_1^n| |\frac{x}{x_1}|^n < |\frac{x}{x_1}|^n$

If  $r = |\frac{x}{x_1}| < 1$ , then we compare  $\sum_{n=n_0}^{\infty} |a_n x^n| \leq \sum_{n=n_0}^{\infty} r^n = \frac{r^{n_0}}{1-r} < \infty$

So  $\sum_{n=0}^{\infty} |a_n x^n| = |a_0| + |a_1| + \dots + |a_{n_0-1}| + \underbrace{\sum_{n=n_0}^{\infty} |a_n x^n|}_{\text{also converges!}}$  convergent

Conclusion: the series converges absolutely in  $(-|x_1|, |x_1|)$ .

(2) We know  $\sum_{n=0}^{\infty} a_n x_1^n$  diverges. We argue by contradiction.

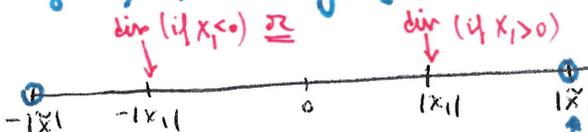
If we can find some  $\tilde{x}$  with  $|\tilde{x}| > |x_1|$  &  $\sum |a_n \tilde{x}^n|$  convergent then we could compare  $0 \leq \sum_{n=0}^{\infty} |a_n| |x_1|^n < \underbrace{\sum_{n=0}^{\infty} |a_n| |\tilde{x}|^n}_{\text{convergent by assumption}} < \infty$

So  $\sum_{n=0}^{\infty} |a_n| |x_1|^n$  would also be convergent (by comparison), and so

$\sum_{n=0}^{\infty} a_n x_1^n$  will converge (abs convergence implies convergence!) This is

a contradiction to our assumption on  $x_1$ . Way out:  $\forall$  all  $\tilde{x}$  with  $|\tilde{x}| > |x_1|$  the series  $\sum_{n=0}^{\infty} |a_n \tilde{x}^n|$  diverges! But we wanted to show  $\sum_{n=0}^{\infty} a_n \tilde{x}^n$  diverges!

Easier: If  $\sum_{n=0}^{\infty} a_n x_1^n$  diverges but  $\forall$  some  $\tilde{x}$  with  $|\tilde{x}| > |x_1|$  the series converges, then by (1) the series  $\sum_{n=0}^{\infty} a_n x_1^n$  would converge absolutely



This is a contradiction!

Conclusion:  $\sum_{n=0}^{\infty} a_n x^n$  diverges

Example (3): (1)  $\forall x$   $x_1 = 1 \implies$  abs conv on  $(-1, 1)$   $\forall$  all  $x$  with  $|x| < 1$ .  
(2)  $\forall x$   $x_1 = -1 \implies$  diverges on  $(-\infty, -1) \cup (1, \infty)$ .  $\square$