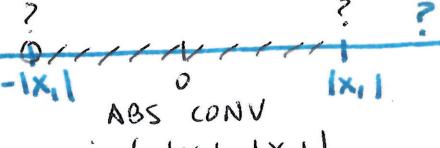


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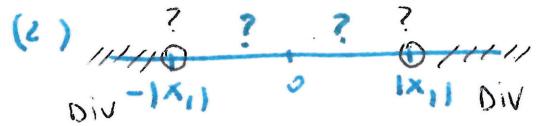
Lecture LIII : §14.2 (cont.) The interval of convergence
 §14.3 Differentiation & Integration of Power Series

Recall: $\sum_{n=0}^{\infty} a_n x^n$ power series is a function of x , defined at $x=0$ & perhaps elsewhere (need convergence of the series for the given choice of x).

Ex.: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ def in $(-1, 1]$

Lemma (last time) (1) 

If convergent for some x_1 , then abs conv in $(-1-x_1, 1-x_1)$

(2)  If divergent for x_1 , then diverges for all x with $|x| > |x_1|$.

Q: What are the possible domains of a power series?

↔ intervals of convergence = region of validity.

- (1) Converges at $x=0$ always ($\text{sum} = a_0$)
- (2) Then, we can grow out until we hit our 1st pt where we diverge
- (3) Then, from then on, we diverge
- (2') If we can't find a 1st point, either we converge absolutely everywhere (so no diverge pts!) or the series only converges at $x=0$.

THM: Fix $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ a power series. Then, precisely

one of the following occurs:

- (1) The series converges only for $x=0$
- (2) The series is absolutely convergent for all x .
- (3) There is a positive real number R (radius of convergence) for which the series: . converges absolutely for $|x| < R$
 . diverges for $|x| > R$

(Nothing can be said for $x = \pm R$, we need to treat these 2 pts separately)

[(1) $R=0$, (2) $R=+\infty$, (3) $0 < R < \infty$]

Intervals of convergence: (1) $[0]$

(2) $R=(-\infty, \infty)$

(3) 4 options: $[-R, R]$, $(-R, R)$,
 $[-R, R)$, $(-R, R]$

Examples: (1) $\sum_{n=0}^{\infty} n! x^n \quad R=0$

Ratio Test $\frac{|(n+1)! x^{n+1}|}{|n! x^n|} = |x| \frac{(n+1)}{n} \xrightarrow{n \rightarrow \infty} |x|$ to all $x \neq 0$. So no abs. convergence for $x \neq 0$.

(2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Ratio Test gives abs. convergence everywhere!

(3) (i) $(-R, R)$: $\sum_{n=0}^{\infty} x^n \quad (R=1)$.

(ii) $(-R, R]$: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (R=1)$

(iii) $[-R, R)$: $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$.

Ratio Test $\left| \frac{A_{n+1}}{A_n} \right| = \frac{|x|^{n+1}}{|x|^n} = |x| \frac{n}{n+1} \xrightarrow{n \rightarrow \infty} |x|$

If $|x| < 1$ we get abs. convergence. } we conclude $R=1$.

If $|x| > 1$ — divergence (of abs. series) } $\xrightarrow{\text{Int of Conv.}} [-1, 1]$

If $x=1$: $\sum_{n=1}^{\infty} \frac{1}{n+1}$ harmonic, so diverges

If $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2 - 1$ (converges by AST)

(iv) $[-R, R]$: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ \Rightarrow find region for abs. convergence.

Use Ratio Test: $\left| \frac{A_{n+1}}{A_n} \right| = |x| \frac{n^2}{(n+1)^2} \xrightarrow{n \rightarrow \infty} |x|$.

If $|x| < 1$; we get abs. convergence } $\Rightarrow R=1$

If $|x| > 1$, — divergence (of abs. series)

If $x=1$: $\sum \frac{1}{n^2}$ converges

If $x=-1$: $\sum \frac{(-1)^n}{n^2}$.. absolutely so it converges also!

Note: We Used Ratio Test (Root Test) to find the radius of convergence.

Q: What if we have $\sum_{n=0}^{\infty} a_n \underbrace{(x-c)^n}_{=z^n}$?

If Radius of Conv. for $\sum_{n=0}^{\infty} a_n z^n$ is $\frac{R}{2}$, same is true for $\sum_{n=0}^{\infty} a_n (x-c)^n$,

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We get abs. conv. in $(c-R, c+R)$
 divergence in $\{x > c+R \text{ or } x < c-R\}$.

End points? If Int. for x is $\left[(-R, R]\right]$, then Int. for x is $\left[(c-R, c+R]\right]$.

§2 Differentiation & Integration

Modelled on polynomials, we ask:

Q: Can we differentiate / integrate power series term-by-term?

A: YES, but the analysis is very delicate (need "uniform convergence" A(S))

THM: Pick $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$ & write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ as ∞ -function in $(-R, R)$. Then:

(1) f is continuous in $(-R, R)$

(2) f is differentiable in $(-R, R)$ & we get f' by term-by-term differentiation

$$f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for all } x \text{ in } (-R, R)$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

(3) f can be integrated term-by-term in $(-R, R)$.

$$\int_0^x f(t) dt = \int_0^x \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n t^{n+1}}{n+1} \Big|_0^x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}.$$

Note: Can make the process in (2) "swallow its tail" and see that $f(x) = a_0 x + a_1 \frac{x^2}{2} + \dots$

$f(x)$ is infinitely differentiable in $(-R, R)$ & $\frac{d^k f}{dx^k} = \sum_{n=k}^{\infty} \frac{n(n-1)\dots(n-k+1)a_n x^{n-k}}{n!}$.

We can use this result to get new expressions for series.

Example ① $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ for x in $(-1, 1)$ abs. conv.

$$\text{We integrate & get } \ln(1+x) = \int_0^x \sum_{n=0}^{\infty} (-t)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This equality is valid in $(-1, 1)$.

Can we push it to $x = \pm 1$? $\begin{array}{lll} \text{For } x=1 & \ln(2) \text{ vs } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots & \text{WORKS!} \\ \text{For } x=-1 & -\infty \text{ vs } -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots = -\infty & \text{DIVERGENCE} \end{array}$

(2) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$ (Ratio Test). Value at $x=0$ is 1.

Can compute f' term-by-term:

$$f' = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

So $f(x)$ is a solution to the diff'l equation $f' = f$ w/ initial condition $f(0) = 1$.

We know another solution: $f = e^x$.

By uniqueness: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ & this is valid in all \mathbb{R} .